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FRACTAL REPRESENTATION OF THE ATTRACTIVE LAMINATION OF AN AUTOMORPHISM OF THE FREE GROUP

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Abstract. In this paper, we extend to automorphisms of free groups some results and constructions that classically hold for morphisms of the free monoid, i.e., so-called substitutions. A geometric representation of the attractive lamination of a class of automorphisms of the free group (irreducible with irreducible powers (*iwip*) automorphisms) is given in the case where the dilation coefficient of the automorphism is a unit Pisot number. The shift map associated with the attractive symbolic lamination is, in this case, proved to be measure-theoretically isomorphic to a domain exchange on a self-similar Euclidean compact set. This set is called the central tile of the automorphism, and is inspired by Rauzy fractals associated with Pisot primitive substitutions. The central tile admits some specific symmetries, and is conjectured under the Pisot hypothesis to be a fundamental domain for a toral translation.

Keywords: *iwip* automorphism, outer automorphism, train-track, lamination, self-affine tiling, Rauzy fractal, substitution, word combinatorics, substitutive dynamical system, graph-directed iterated function system, Pisot number, discrete spectrum.

Introduction

Symbolic dynamical systems were first introduced, more than a century ago, to gain insight into the dynamics of some geometric maps. The simplest of them, i.e., the subshifts of finite type, proved particularly well adapted to studying dynamical systems for which past and future are disjoint, e.g., toral automorphisms or Pseudo-Anosov diffeomorphisms of surfaces [33].

Geometric models of substitutions. Of interest are also self-similar systems, which can be loosely defined as systems where the large-scale recurrence structure is similar to the small-scale recurrence structure, or more precisely as systems which are topologically conjugate to their first return map on a particular subset. They also have natural models, namely symbolic systems associated with substitutions, defined as morphisms of the free monoid (a substitution consists in replacing a letter in a finite alphabet by a nonempty finite word on this alphabet) [42, 41]. A basic idea is that, as soon as self-similarity appears, a substitution is hidden behind the original dynamical system.

Conversely, substantial literature is devoted to the geometric interpretation of substitutions, as domain exchanges acting on a self-affine structure [43, 51, 18, 19, 47, 48, 49, 41], as numeration systems [23, 24, 13], or as expanding foliations in space tilings [11]. All these representations are based on the minimal symbolic dynamical system generated by the substitution σ, which we assume from now on to be primitive, i.e., there exists an iterate σ^n such that the image of any letter under σ^n contains all the other letters.

Substitutive dynamical systems were first analyzed in particular cases in [43] and [51]. These constructions were soon afterwards generalized to irreducible Pisot substitutions: a primitive substitution is said to be irreducible Pisot if all its eigenvalues except its dominant one have modulus non-zero and strictly smaller than 1; the dominant eigenvalue is then a Pisot number. The substitution is said to be unit Pisot when this Pisot dominant eigenvalue is a unit integer. In many cases it is possible to associate with an irreducible unit Pisot substitution a finite family of compact

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subtiles: the union of these tiles is called (according to the authors) central tile, Rauzy fractal, or Thurston tile, and it provides a geometric representation of the substitutive dynamical system. The subtiles are solutions of a self-affine graph-directed Iterated Function System [37] and generate both a periodic and a self-replicating multiple tiling. In all known examples, these multiple tilings are in fact tilings. This is equivalent to the fact that the substitutive dynamical system has pure discrete spectrum according to [11]: the dynamical system is indeed measure-theoretically isomorphic with a translation on the $(d-1)$-dimensional torus $\mathbb{T}^{d-1}$, the central tile being a fundamental domain for $\mathbb{T}^{d-1}$, where we denote by d the degree of the dominant eigenvalue, that is, the cardinality of the alphabet of the substitution. A combinatorial necessary and sufficient condition for this pure discrete spectrum is stated in terms of the so-called conditions of geometrical coincidences or of super-coincidences [31, 11]. The discrete spectrum conjecture, which resists now for several years, asserts that all unimodular Pisot substitution dynamical systems have a discrete spectrum. This also has consequences for the effective construction of Markov partitions for toral automorphisms, whose main eigenvalue is a Pisot number [30, 40, 46]. We recall in more details the construction of the central tile of a Pisot substitution in Section 3.1.

Geometric models of free group automorphisms. In this paper, we intend to investigate how this picture can be completed when extending from substitutions (morphisms of the free monoid) to morphisms of the free group. Any substitution naturally extends to an endomorphism of the free group, and those that extend to an automorphism are called *invertible* substitutions. Although invertibility does not play a significant role in the case of substitutions, it does in the case of morphisms of the free group; indeed the general theory of endomorphisms is not yet well understood, and most geometric constructions lead to automorphisms. Here we thus focus solely on the case of automorphisms of the free group. In this framework, substitutions appear as a special family of morphisms, namely *positive* morphisms.

Another very specific (and quite well known) family of morphisms of the free group is provided by *geometrical* automorphisms induced by homeomorphisms of orientable surfaces with non-empty boundary. Then the homeomorphism of the surface can be coded into an automorphism of the homotopy group of the surface. The case of the free group of rank 2 is special: all automorphisms of the group F_2 are geometrical. On the contrary, the so-called Tribonacci automorphism $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$ is not geometrical; more generally, most automorphisms of the free group are not geometric. As we shall see in Appendix 2, no irreducible automorphism on a free group of odd rank comes from an homeomorphism of an orientable surface.

The dynamical behavior of automorphisms of a free group is much more difficult to understand than the behavior of substitutions. The problem is that the study of substitutions is based on the existence of a finite set of infinite words which are fixed or periodic under the action of the substitution. Such infinite words clearly exist, because one can always find a letter a and a power σ^p of the substitution such that $\sigma^p(a)$ begins with a; hence the words $\sigma^{np}(a)$ are prefixes of each other, and tend, under weak conditions, to an infinite word fixed by σ^p. The iterate of any word under the substitution then converges to this finite set of periodic points. If we replace the free monoid by the free group, this does not work any more, or at least not in such a simple way, due to the appearance of cancellations. An elementary example is that of a conjugacy automorphism, $i_w : a \mapsto waw^{-1}$, which fixes the word w.

An impressive achievement was obtained by Bestvina, Feighn and Handel [14]: they give a good representative for any automorphism, called an improved relative train-track map, which takes care of cancellations. The simplest dynamically nontrivial automorphisms are irreducible ones, with irreducible power (*iwip*): they are the algebraic equivalent of pseudo-Anosov homeomorphisms of surfaces. For such automorphisms, using train-track representatives [17], Bestvina, Feighn and Handel show that one can find a reduced two-sided recurrent infinite word on which the automorphism acts without cancellation and which is fixed by some power of the automorphism. Then they construct a dynamical system, called the attractive symbolic lamination [15], by taking the closure of the orbit of this word under the shift map.

Since then, a general theory of laminations of free (or hyperbolic) groups has been put forward [22]. It appears that these laminations have two presentations, namely an algebraic one and a symbolic one. Algebraically, a lamination is a set of geodesic lines in the free group which is closed (for the topology induced by the boundary topology), invariant under the action of the group and flip-invariant (i.e., orientation-invariant). Symbolically, a lamination is a classic symbolic dynamical system: it is a shift and flip-invariant closed set of two-sided sequences.

In this paper, we detail how a central tile can be used to represent the attractive lamination, in the particular case of a unit Pisot dilation coefficient. The map $x \mapsto x^{-1}$ produces symmetries in the central tile. This allows us to recover the dynamics of the attractive lamination of the free group automorphism as a first return map into a specific self-similar compact set. Moreover, this also allows us to prove in some specific cases that the symbolic dynamical system describing the attractive lamination has a pure discrete spectrum, providing a spectral interpretation of the central tile.

Note that a basic difference between the free monoid and the free group is that the free monoid has a canonical basis, which is not the case for the free group. Hence, although the attractive lamination is intrinsic, there exist several symbolic codings for it. It seems that deciding to choose a specific coding, hence a specific symbolic dynamical system, corresponds in particular to choosing a discrete time to move on the leaf of the formal lamination; it would be interesting to find an intrinsic way to define a flow on the lamination; the different codings would then appear as first-return map of this flow to a section, in the spirit of [11].

Principle of the construction. In this paper, we focus on the case where the dilation coefficient of the *iwip* automorphism φ is a unit Pisot number β.

We take a primitive train-track representative f of φ, acting on a graph with k edges. We can consider f as a morphism of the free group on k letters. We use the fact that f has no cancellation under the iteration on an edge to consider a letter e_i and its inverse e_i^{-1} as different letters, so that f gives rise to a substitution σ_f (that we call double substitution) on alphabet $\mathcal{A}$ consisting of $2k$ letters $\{e_1 \ldots, e_k, e_1^{-1}, \ldots, e_k^{-1}\}$.

Let $\mathbf{l}$ denote the abelianization map. This substitution σ_f has obvious symmetries. If we consider its abelianization $\mathbf{l} \circ \sigma_f$, it preserves the space generated by the vectors $\mathbf{l}(e_i) - \mathbf{l}(e_i^{-1})$, and its action on this space is given by the abelianization of f. It also preserves the space generated by the vectors $\mathbf{l}(e_i) + \mathbf{l}(e_i^{-1})$, and its action on this space has the dilation coefficient β of φ. There is an invariant subspace associated with β and its conjugates, which splits into an eigenline associated with β and an invariant (contracting) subspace $\mathbb{H}_c$ associated with the conjugate eigenvalues.

The principle of the construction is then to associate a broken line in $\mathbb{R}^{2k}$ with a periodic point of f, and to project the vertices of this broken line on $\mathbb{H}_c$ along its natural complement. This set is bounded in $\mathbb{H}_c$, and its closure is a compact set, called the central tile of f. We note that there is a particular case, the orientable case, where the train-track f is a substitution up to replacement of some letters by their inverses. In this case, the central tile $\mathcal{T}_f$ decomposes into two distinct symmetric parts. If this is not the case (nonorientable case), the substitution σ_f is primitive, and we have:

Theorem. *Let $f : G \to G$ be a nonorientable train-track for a unit Pisot* iwip *outer automorphism Φ. Let k be the number of edges of G. Let d be the degree of the dilation coefficient of f (or Φ). The central tile $\mathcal{T}_f$ of f is a compact subset with a nonempty interior, hence non-zero measure, of a $(d-1)$-dimensional subspace of $\mathbb{R}^{2k}$. It is divided into $2k$ subtiles, namely $\mathcal{T}_f(e)$ and $\mathcal{T}_f(e^{-1})$ for any edge e of G, that satisfy a graph-directed Iterated Function System.*

This central tile has remarkable properties. We denote by $L_\Phi{}^+(G)$ the attractive symbolic lamination in G-coordinates given by the nonorientable train-track $f : G \to G$ (for more details, see Definition .3 below).

Theorem. *Let $f : G \to G$ be an nonorientable train-track for a unit Pisot* iwip *outer automorphism Φ. We furthermore assume that the double substitution σ_f satisfies the so-called combinatorial condition of strong coincidence. Then there exists a domain exchange E_f acting on the central tile $\mathcal{T}_f$ which is defined almost everywhere, and which acts on the k pieces $\mathcal{T}_f(e) \cup \mathcal{T}_f(e^{-1})$. The attractive symbolic lamination in G-coordinates $L_\Phi{}^+(G)$ provided with the shift map S is measure-theoretically isomorphic to $(\mathcal{T}_f, E_f)$, that is, there exists a map $\mu : L_\Phi{}^+(G) \to \mathcal{T}_f$ that is continuous, onto and almost everywhere one-to-one, and that satisfies $\mu \circ S = E_f \circ \mu$.*

This implies the following properties of the central tile.

Theorem. *The central tile of a unit Pisot* iwip *nonorientable train-track f is symmetric with respect to the origin: $\mathcal{T}_f = -\mathcal{T}_f$. Furthermore, the subtiles $\mathcal{T}_f(e)$ and $\mathcal{T}_f(e^{-1})$ are pairwise symmetric.*

This is reminiscent of the properties of measured foliations of surfaces. Orientable foliations can be studied by interval exchanges, which give rise to a special type of symbolic dynamics. This symbolic dynamical system has no reason to be stable under reversal, since the interval exchange has no need to be conjugate to its inverse [5]. General nonorientable foliations cannot be directly studied in this way. However, one can always lift to an orientation cover to obtain an orientable foliation; but in this case, this orientable foliation admits by construction an orientation-reversing symmetry, and a symbolic dynamical system invariant under reversal.

The formalism introduced here extends in a natural way to non-unit Pisot dilation coefficients following the approach of [47, 12]; in this latter case, the central tile has p-adic components. Central tiles for automorphisms of free groups in the non-Pisot case have also been considered in [25]. Let us finally quote [8], where a simple example of an automorphism of the free group on 4 generators, with an associated matrix that has 4 distinct complex eigenvalues, two of them of modulus larger than 1, and the other 2 of modulus smaller than 1 (non-Pisot case) is handled in details; the link with the present construction is unclear.

Outline of the paper. Section 1 introduces the basic concept of our study, namely *iwip* automorphisms. More precisely, Section 1.1 recalls basic notions on substitutive dynamical systems, whereas Section 1.2 deals with automorphisms of free groups. The notions of topological representatives of an outer automorphism and of train-track maps are introduced in Section 1.3.

Section 2 is devoted to the definition of the attractive symbolic lamination of an *iwip* automorphism with a train track map.

Section 3.1 recalls the construction of central tiles associated with Pisot substitutions. In Section 3.2, we define the main topic of study of the present paper, namely the central tile associated with *iwip* automorphisms. The essential ingredient of our construction, introduced in Section 2, is the double substitution associated with a topological representative of the outer automorphism which is a train-track map, defined by duplicating the alphabet. This substitution generates the central tile in the usual substitutive sense. We consider the topological and geometric properties of the central tile in Section 3.3. We then introduce, in Section 3.4, a first dynamical system acting on them and defined as an exchange of pieces, as well as multiple tilings of the space whose prototiles are given by the central tiles. This allows us, in Section 3.5, to give a geometric and spectral interpretation of the dynamical system associated with the attractive lamination.

We give several examples in Section 4 (some examples are given before to illustrate the definitions and theorems).

We conclude this paper by evoking open questions and further research work in Section 5.

Appendix 1 is meant to complete the picture by showing that the attractive lamination has an intrinsic definition; although the object we deal with depends on several noncanonical choices, there are underlying intrinsic objects. Appendix 2 shows that some automorphisms do not come from homeomorphisms of orientable surfaces.

1. Substitutions and automorphisms of the free group

1.1. **Substitutive symbolic dynamical systems.** We recall in this subsection the definition of the symbolic dynamical system associated with a primitive substitution. For further details, the reader is referred to [42, 34, 41].

Monoid, sequences, language. Let $\mathcal{A} = \{a_1, \ldots, a_N\}$ be a finite set called alphabet whose elements are called letters. The *free monoid* $\mathcal{A}^*$ on the alphabet $\mathcal{A}$ with empty word ε is defined as the set of finite words on the alphabet $\mathcal{A}$, that is, $\mathcal{A}^* := \{\varepsilon\} \cup_{i\in\mathbb{N}} \mathcal{A}^i$, endowed with the concatenation map. We denote by $\mathcal{A}^{\mathbb{N}}$ (resp. $\mathcal{A}^{\mathbb{Z}}$) the set of one-sided (resp. two-sided) sequences on $\mathcal{A}$. The topology of the set of one sided-sequences or two-sided sequences is the product topology of the discrete topology on each copy of $\mathcal{A}$; it is metrizable.

The *length* $|w|$ of a word $w \in \mathcal{A}^*$ is defined as the number of letters it contains. For any letter $a \in \mathcal{A}$, we denote by $|w|_a$ the number of occurrences of a in w. Let $\mathbf{l} : w \in \mathcal{A}^* \mapsto (|w|_a)_a \in \mathbb{N}^n$ be the natural homomorphism obtained by abelianization of the free monoid, called the *abelianization map.*

A *factor* of a sequence x (which is assumed to be either one-sided or two-sided) or a word x is a finite word $w = w_1 \cdots w_n \in \mathcal{A}^*$ that occurs in x, i.e., there exists j such that $x_j = w_1, \cdots, x_{j+n-1} = w_n$. The *language* of the sequence x is the set of finite factors of x.

Symbolic dynamics, minimality and quasiperiodicity. The *shift map* S on the sets $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ is defined by $S(u_i) = (v_i)$ with $v_i = u_{i+1}$ for all i. A one-sided (resp. two-sided) symbolic dynamical system is a pair (X, S), where S is the shift, and X is a closed subset of $\mathcal{A}^{\mathbb{N}}$ (resp. $\mathcal{A}^{\mathbb{Z}}$) which is invariant by the shift.

In particular, with any sequence $w = (w_n)_{n\in I}$ (assumed to be either one-sided or two-sided), one can associate the symbolic dynamical system (X_w, S) where X_w is defined as the closure of the orbit of w under the action of the shift map: $X_w = \overline{\{S^n w;\ n \in I\}}$. The set X_w consists of all sequences whose language is contained in that of w; in particular, it is completely defined by the language of w.

A one-sided or two-sided sequence w is said to be *quasiperiodic* or *repetitive* when it satisfies the *bounded gap property.* Every factor of w occurs an infinite number of times in w, with bounded recurrence time, or equivalently, for all $K > 0$, there exists $C > 0$ such that every factor of w of length C contains all factors of w of length K. When the sequence w is quasiperiodic, then the symbolic dynamical system X_w is *minimal*, i.e., every nonempty shift-invariant closed subset of X_w is equal to X_w. For further details, see for instance [42].

Substitutions. A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$ such that the image of each letter of $\mathcal{A}$ is nonempty; to avoid trivial cases (projection or permutations of letters), we will always suppose that for at least one letter, say a, the length of the successive iterations $\sigma^n(a)$ tends to infinity. A substitution naturally extends to the set of one-sided or two-sided sequences $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$. We associate with every substitution σ its *incidence matrix* (or *transition matrix*) $\mathbf{M}_\sigma$ which is the N-square matrix obtained by abelianization, i.e., $\mathbf{l}(\sigma(w)) = \mathbf{M}_\sigma \mathbf{l}(w)$ for all $w \in \mathcal{A}^*$. We now define the important property of *Primitivity.*

Definition 1.1. A substitution σ is *primitive* if there exists an integer K (independent of the letters) such that, for each pair $(a,b) \in \mathcal{A}^2$, the word $\sigma^K(a)$ contains at least one occurrence of the letter b.

Remark 1. There exists a notion of primitivity for square matrices with nonnegative integral entries: such an N-square matrix $\mathbf{M}$ is said to be *primitive* if there exists an integer $K > 0$ such that for all (i,j) $(1 \leqslant i,j \leqslant N)$, the (i,j)-entry of $\mathbf{M}^K$ is positive. It is said to be *irreducible* if for all (i,j), there exists an integer $K = K(i,j) > 0$ such that the (i,j)-entry of $\mathbf{M}^K$ is positive. In fact, the matrix $\mathbf{M}$ is primitive if, and only if, all the positive powers of $\mathbf{M}$ are irreducible [45]. Note that a substitution is primitive if, and only if, its incidence matrix $\mathbf{M}_\sigma$ is primitive.

A one-sided *periodic point* of the substitution σ is an infinite word $w = (w_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma^\nu(w) = w$ for some $\nu > 0$. A two-sided *periodic point* of the substitution σ is an infinite word $w = (w_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satisfies $\sigma^\nu(w) = w$ for some $\nu > 0$, and furthermore its central pair of letters $w_{-1}w_0$ belongs to the image of some letter by σ^n. All substitutions admit periodic points; in the case of primitive substitutions, one can say more: the number of one-sided (resp. two-sided) periodic points is bounded by the size of the alphabet (resp. by its square). Indeed, one can always, by a simple argument, find a letter a and a power σ^n of σ such that $\sigma^n(a)$ begins with a; then the words $\sigma^{jn}(a)$ are prefixes of each other, and determine a periodic point, the unique sequence which admits all the words $\sigma^{jn}(a)$ as prefixes. The same proof shows that any one-sided fixed point is completely determined by its first letter, because, by primitivity, for any letter a, the length of the successive iterations $\sigma^n(a)$ tends to infinity. It is also clear that any periodic two-sided sequence is determined by its central pair of letters.

One can say more, namely if u is a periodic point for a primitive substitution σ, the set X_u does not depend on u, but only on σ, since, by primitivity, all periodic points have the same language. This leads to the following definition:

Definition 1.2. The *symbolic dynamical system generated by a primitive substitution* σ is the system (X_u, S), where u is any periodic point of σ; this will be denoted by (X_σ, S) .

The system (X_σ, S) is *minimal* and *uniquely ergodic*, i.e., there exists a unique shift-invariant probability measure μ_{X_σ} on X_σ [42].

Remark 2. One could ask whether we considered one-sided or two-sided fixed points in Definition 1.2; this is actually not very important, since one proves that the one-sided system is one-to-one, except for a finite number of points with a finite number of preimages (see e.g., [41], Chap. 5), hence the natural projection of the two-sided system on the one-sided system is one-to-one, except on a countable number of points; in particular, the two systems are measurably isomorphic. The one-sided system lends itself to an elementary presentation; but the two-sided system has better properties (in particular, it is one-to-one by construction).

1.2. **Automorphisms of free groups.** A substitution σ defined over the free monoid $\mathcal{A}^*$ associated with a finite alphabet $\mathcal{A}$ naturally extends to the free group on $\mathcal{A}$, by defining $\sigma(a^{-1}) = \sigma(a)^{-1}$. Hence, it is natural to try to extend the previous construction to a general endomorphism of the free group F_N on N generators. (As we said in the introduction, we will only consider here automorphisms of the free group, since the general theory of endomorphisms is not developed enough for our purposes.)

But two problems arise. The first one is that cancellations may appear when we extend from the free monoid to the free group. This changes things even for substitutions: the Sturmian substitution (see e.g., [34]) $a \mapsto aba \quad b \mapsto ba$ fixes no nonempty word in the free monoid, but it fixes the commutator $aba^{-1}b^{-1}$! This implies that the previous simple construction of a fixed infinite word by iterating the image of a letter does not directly apply to the general case of an automorphism. In general, cancellations appear, and we will not get an increasing sequence of prefixes $\sigma^{jn}(a)$. For example, as noted in the introduction, if we consider the conjugacy by a word w (also called *inner automorphism*), $i_w : g \mapsto wgw^{-1}$, we have $i_w(w) = w$, hence we cannot get an infinite periodic point by iterating on w. We thus need a condition that generalizes the primitivity condition defined earlier, and allows us to define an infinite fixed point when we start from a letter. We define below the so-called *iwip* automorphisms, as introduced in [17, 14] (Definition 1.3). The *iwip* condition appears to be a generalization of the primitivity condition.

The second problem is of a more theoretical nature, namely the basis (alphabet) used to build a free monoid is canonical, while it is not for the free group. More generally, in many situations, automorphisms of the free groups arise as automorphisms of the fundamental group of a graph, induced by a continuous map of the graph to itself, as we shall see in Section 1.3. However, such an automorphism is only defined up to a composition by an inner automorphism (due to the choice of an arbitrary path from the basis point of the fundamental group to its image). The difficult question, which does not arise for substitutions, is to understand what is intrinsic in our constructions.

In this subsection, we define the notions necessary to deal with these two problems.

Free group, basis and factor. A good introduction to free groups and graphs can be found in [36]. Let $\mathcal{A}_N = \{a_1, \ldots, a_N\}$ be a finite alphabet with N letters. Let $\mathcal{A}_N^{-1} = \{a_1^{-1}, \ldots, a_N^{-1}\}$ denote the inverse letters of $\mathcal{A}_N$. The *free group* F_N generated by $\mathcal{A}_N$ is the quotient of the free monoid $(\mathcal{A}_N \cup \mathcal{A}_N^{-1})^*$ under the congruence relation generated by $a_i a_i^{-1} = a_i^{-1} a_i = \varepsilon$. A finite word or an infinite sequence on the alphabet $\{a_1^{\pm}, \cdots, a_N^{\pm 1}\}$ is said to be *reduced* if it has no factor $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

A *free factor* G of the free group F_N is a subgroup generated by a subset of some basis of F_N; equivalently, a subgroup G of F_N is a free factor if there exists a subgroup H of F_N such that F_N is the free product $G * H$ of G and H [36].

Automorphisms, positive automorphisms and outer automorphisms of free groups. We denote by $\mathrm{Aut}(F_N)$ the *group of automorphisms* of F_N. Specific morphisms of free groups are the *positive* ones. A morphism φ of F_N is said to be *positive* if there exists a basis $\mathcal{A}$ such that the reduced words $\varphi(a)$, for every $a \in \mathcal{A}$, contain no letter with a negative exponent. Equivalently, positive morphisms of a free group F_N are extensions of substitutions on a monoid $\mathcal{A}$ generating F_N.

The set $\mathrm{Inn}(F_N)$ of all inner automorphisms (or conjugacy) is a normal subgroup of $\mathrm{Aut}(F_N)$. The quotient group of $\mathrm{Aut}(F_N)$ by $\mathrm{Inn}(F_N)$ is denoted by $\mathrm{Out}(F_N)$; its elements are called *outer automorphisms* of F_N.

Remark 3. A substitution is called *invertible* when it extends to an automorphism of the free group. The two-letter invertible substitutions are exactly Sturmian substitutions (see for instance [34]); they have been widely studied and have a very rich combinatorial characterization as Sturmian substitutions (see references in [34]). For more than two letters, the set of invertible substitutions forms a monoid which seems to have a more complicated structure (see [50]).

Iwip *automorphisms.* In this article, we will focus on a special class of automorphisms of free groups, namely the *iwip* automorphisms. As mentioned in the introduction, they are the natural equivalent of primitive substitutions, or of pseudo-Anosov homeomorphisms of surfaces. In the next subsection, we will describe why this analogy with primitivity holds. Let us give their definition.

Definition 1.3. An automorphism $\varphi \in \mathrm{Aut}(F_N)$ is said to be *iwip*, that is, irreducible with irreducible powers, if no proper free factor F of F_N is mapped by any positive power of φ to a conjugate of F. An outer automorphism $\Phi \in \mathrm{Out}(F_N)$ is *iwip* if one (and hence any) automorphism $\varphi \in \mathrm{Aut}(F_N)$ in Φ is *iwip*.

1.3. **Train-track representative of an *iwip* automorphism.** A first idea to control cancellations and gain insight into the dynamics of an automorphism is to choose an appropriate basis of the free group; but this is not enough for our purposes. In this section, we introduce a much more powerful tool based on [17]. It consists in using graphs and their homotopy equivalences to reduce the cancellations occurring in iterations as much as possible.

Combinatorial and topological graphs. We consider a *combinatorial graph* G, given by a finite set of vertices $\mathcal{V} = \{v_1, \dots v_m\}$ and a finite set of oriented edges $\mathcal{E}$ provided with a map $\rho : \mathcal{E} \to \mathcal{V} \times \mathcal{V}$ mapping an edge to its initial and terminal vertices (note that this definition allows graphs to contain several edges joining the same vertices). The set of edges $\mathcal{E}$ is supposed to be symmetric: for every edge $e \in \mathcal{E}$, there exists an opposite edge in $\mathcal{E}$, denoted by e^{-1}, with an opposite orientation (in particular, $\rho(e^{-1}) = (b, a)$ if $\rho(e) = (a, b)$).

This combinatorial graph definition is not precise enough to tackle the problem with algebraic topology and geometry. We need to introduce a topological representation of graphs, i.e., the notion of one-dimensional CW-complex [36]. The *topological graph associated with a combinatorial graph* is defined as follows.

- For each pair $\{e_i, e_i^{-1}\}$ of opposite edges, chose a *canonical representative* (an orientation of the edge).
- For every canonical representative e_j, let $I_j = [a_j, b_j]$ be a segment (topological space homeomorphic to a compact interval in $\mathbb{R}$).
- Define an equivalence relation $\sim$ on the disjoint union $\mathcal{V} \coprod \left(\coprod_{j=1}^{k} I_j\right)$ by taking the transitive closure of $a_i \sim v_j$ if e_i has v_j as origin point, and $b_i \sim v_j$ if e_i has v_j as end point.
- The *topological graph* associated with the graph G is the quotient of $\mathcal{V} \coprod \left(\coprod_{j=1}^{k} I_j\right)$ by this equivalence relation, endowed with the quotient topology.

Conversely, the combinatorial graph with which the topological graph is associated is uniquely determined. In the following, we will use both a combinatorial and a topological point of view. When there is no ambiguity, we will simply call them graphs.

The *valence* of a vertex of a topological graph is the number of extremities of canonical edges attached to this vertex; more formally, it is the number of preimages of a vertex by the canonical projection $\coprod_{j=1}^{k} I_j \to G$. A graph is said to be *trivial* if its components are vertices. A *tree* is a contractible connected graph. A graph is a *forest* if its components are trees.

Paths in a graph. Let G be a combinatorial graph with set of edges $\mathcal{E}$. A *path* in G indexed by a set I is a sequence of edges $w = (e_i)_{i \in I} \in \mathcal{E}^I$ such that:

- the notation I for the set of indices is either a finite set $I = \{1, \dots n\}$ (then the path is said to be finite), or equal to $\mathbb{N}$ (one gets a one-sided sequence that is called a *ray*), or equal to $\mathbb{Z}$ (one gets a two-sided sequence that is called a *line*);
- w is the coding of a walk in the graph, that is, for all i, the initial vertex of e_{i+1} is equal to the terminal vertex of e_i;
- w is *reduced*, that is, $e_i \neq e_{i+1}^{-1}$ for all $i \in I$.

In a topological graph, a path is an immersion (that is, a locally injective map) of a segment $[a, b]$ (for a finite path), or of $[0, +\infty]$ (for a ray), or of $\mathbb{R}$ (for a line), or a point (for a trivial path).

Note that, by definition, for us, and following [14], paths are always reduced, even though this is not standard terminology.

Marked graph and topological representative. The *rose with N petals*, denoted by R_N, is the graph with a unique vertex $*$ and N edges (from $*$ to $*$). We identify its fundamental group $\pi_1(R_N, *)$ with F_N: a basis of F_N is given by the edges [36].

Recall that a homotopy equivalence $f : G \to G'$ between two topological graphs is a continuous map inducing an isomorphism on the fundamental groups of G and G', that is, a continuous map such that there exists a continuous map $g : G' \to G$ so that $f \circ g$ and $g \circ f$ are homotopic to the identity. For any connected topological graph, there is a unique number N (the *cyclomatic number* of the graph G, equal to $1 - \chi(G)$, where $\chi(G)$ is the Euler characteristic of the graph, see also Remark 4) such that there exists a homotopy equivalence $\tau : R_N \to G$. A topological graph G provided with such a homotopy equivalence (called the *marking*) $\tau : R_N \to G$ is a *marked graph*. Considering that τ induces the identity between $\pi_1(R_N, *)$ and $\pi_1(G, \tau(*))$, we identify F_N (in a non-canonical way) with $\pi_1(G, \tau(*))$. Note that a connected graph G is a tree if, and only if, its cyclomatic number is 0.

Let G be a marked graph with marking $\tau : R_N \to G$, and a homotopy equivalence $f : G \to G$. Then f naturally induces an automorphism of $\pi_1(G, \tau(*)) \simeq F_N$. This automorphism is defined up to composition by an inner automorphism, since the identification between $\pi_1(G, \tau(*))$ and $\pi_1(G, f(\tau(*)))$ is made up to the choice of an arbitrary path between $\tau(*)$ and $f(\tau(*))$. We call *outer automorphism associated with* f and denote it by Φ the outer class of automorphisms induced by the homotopy equivalence $f : G \to G$.

Let $\Phi \in \mathrm{Out}(F_N)$ be an outer automorphism. A *topological representative* of Φ is a map $f : G \to G$, where G is a marked graph, such that:

- the image of a vertex is a vertex,
- the image of an edge is a finite path of G (as defined above),
- f induces Φ on $F_N \simeq \pi_1(G, \tau(*))$ (in particular, f is a homotopy equivalence).

A subgraph H of G is said to be f-invariant if $f(H) \subset H$. A topological representative $f : G \to G$ is said to be *irreducible* if:

- the graph G has no vertex of valence one,
- the f-invariant subgraphs of G are trivial.

A topological representative $f : G \to G$ is said to be *primitive* if for all $k \in \mathbb{N}$, $f^k : G \to G$ is irreducible.

Train-track map. If f is a topological representative of an outer automorphism, the image of any edge is a path, but the image of a path is not necessarily a path, i.e., cancellations can occur. In particular, any automorphism of F_N with a given basis has a topological representative on the rose with N petals, and these cancellations are the reason why an infinite fixed word cannot be immediately found; this is one motivation for the following definition.

Definition 1.4. A *train-track map* is a topological representative $f : G \to G$ of a free group automorphism, such that:

- G has no vertex of valence 1 or 2,
- for all edge $e \in \mathcal{E}$ and all $n > 0$, $f^n(e)$ is a path in G.

The crucial point of this definition is that no cancellation occurs when iterating f on an edge.

The existence of a train-track map representing a given automorphism is far from obvious. The simplest examples of automorphisms that can be represented by a train-track map are the positive ones: every substitution on N letters is trivially represented by a train-track on the rose R_N. M. Bestvina and M. Handel prove in [17] the following main result:

Theorem 1.5 (Bestvina-Handel)**.** *Every* iwip *outer automorphism $\Phi \in Out(F_N)$ admits a train-track map $f : G \to G$ as a topological representative.*

Moreover, they give an algorithm that produces, from the data of an automorphism $\varphi \in \mathrm{Aut}(F_N)$, a topological representative $f : G \to G$ of the outer automorphism $\Phi \in \mathrm{Out}(F_N)$ defined by φ, which is:

- either a train-track map which is irreducible,
- or a reduction, i.e., G has no vertex of valence one, there is no f-invariant non-trivial forest, and f is not irreducible.

In the second case, f cannot be *iwip*. In fact, using the algorithm of Bestvina-Handel, one can check effectively whether an automorphism is *iwip*.

Transition matrix, dilation coefficient. Let $f : G \to G$ be a topological representative of an outer automorphism $\Phi \in \mathrm{Out}(F_N)$, and denote by $e_1, e_1^{-1}, \dots, e_k, e_k^{-1}$ the edges of G. The *transition matrix* $\mathbf{M}_f$ associated with f is a k-square matrix with (i,j)-entry the number of times $f(e_i)$ crosses e_j or e_j^{-1}. A topological representative is primitive if, and only if, its transition matrix is primitive.

By its definition, $\mathbf{M}_f$ has nonnegative integral entries. If, moreover, $\mathbf{M}_f$ is primitive, then the Perron-Frobenius theorem applies such that $\mathbf{M}_f$ has a unique dominant real eigenvalue $\lambda_f > 1$ called the *Perron-Frobenius eigenvalue.* The unique dominant eigenvector with positive real entries normalized such that the sum of entries is equal to one is denoted by $\mathbf{v}_f = (v_1, \dots, v_n)$. The following is stated in [17]:

Theorem 1.6 (Bestvina-Handel)**.** *If $\Phi \in$ Out(F_N) is an* iwip *outer automorphism, then there exists a primitive topological representative f for Φ whose associated Perron-Frobenius eigenvalue λ_f is minimal (i.e., it is less than or equal to the Perron-Frobenius eigenvalue associated with any primitive topological representative of Φ). Moreover, any such primitive topological representative f for Φ without vertex of valence 2, is a primitive train-track map.*

It follows that λ_f only depends on the outer automorphism Φ: we shall denote it by λ_Φ and call it the *dilation coefficient of* Φ. A length can be given to any finite path w in G by considering that each edge e_i is isometric to an interval of length v_i: we call it the *Perron-Frobenius length*, and denote it by $|\ |_{PF}$. Note that if w is a finite path in G such that no cancellation occurs when calculating $f(w)$, we have: $|f(w)|_{PF} = \lambda_\Phi |w|_{PF}$, which justifies the dilation coefficient name for λ_Φ.

Remark 4. Since the graph G of a train-track map $f : G \to G$ representing a given $\Phi \in \mathrm{Out}(F_N)$ has no vertex of valence 1 or 2, there exist only finitely many such graphs. Indeed, since every vertex has at least valence 3, the number E of edges and the number V of vertices satisfy $2E \geqslant 3V$; but the cyclomatic number N of the graph is equal to $E-V+1$, hence we must have $N = E-V+1 \geqslant \frac{1}{2}V+1$; this implies that $V \leqslant 2(N-1)$, and $E = N+V-1 \leqslant 3N-3$; but the number of combinatorial graphs with a bounded number of edges and vertices is finite. Moreover, since the transition matrix $\mathbf{M}_f$ of a train-track map has nonnegative integral entries and fixed Perron-Frobenius eigenvalue λ_Φ, there exist only finitely many train-track maps representing Φ. Another consequence of this discussion is that the degree of the dilation coefficient of an *iwip* outer automorphism $\Phi \in \mathrm{Out}(F_N)$ is bounded above by $3N-3$ (since it is bounded above by the size of the transition matrix of a train-track map representing Φ).

The following proposition gives a useful sufficient condition for checking the *iwip* property. Let φ be an automorphism of F_N. We define $\mathbf{M}_\varphi^+$ (resp. $\mathbf{M}_\varphi^-$) as the N-square matrix with (i,j)-entry the number of occurences of a_i (resp. a_i^{-1}) in $\varphi(a_j)$. Let $\mathbf{A}_\varphi$ be the *abelianization matrix* defined as $\mathbf{A}_\varphi := \mathbf{M}_\varphi^+ - \mathbf{M}_\varphi^-$. This matrix provides information on the action of φ since $\mathbf{l}(\varphi(w)) = \mathbf{A}_\varphi(\mathbf{l}(w))$ for all $w \in F_N$ (we recall that the abelianization map $\mathbf{l}$ is defined in Section 1.1).

Proposition 1.7. *Let φ be an automorphism of the free group. If the characteristic polynomial of the abelianization matrix $\mathbf{A}_\varphi$ is irreducible over $\mathbb{Z}$, then φ is an* iwip *automorphism.*

Proof. If φ is not an *iwip* automorphism, there exists a proper free factor F of F_N φ-invariant. By abelianization, F maps to a proper sublattice of $\mathbb{Z}^N$ which is $\mathbf{A}_\varphi$-invariant; this contradicts the irreducibility of the characteristic polynomial of $\mathbf{A}_\varphi$ over $\mathbb{Z}$. □

Example 1. Let φ_1 be the automorphism of $F_3 =< a, b, c >$ given by $a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b$. This automorphism is the inverse of the well-studied substitution $\sigma_1 : a \mapsto ab,\ b \mapsto c,\ c \mapsto a$.

One has $\mathbf{M}_{\varphi_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\mathbf{A}_{\varphi_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. The characteristic polynomial of the abelianization matrix $\mathbf{A}_{\varphi_1}$ is equal to X^3+X-1 and it is irreducible. According to Proposition 1.7, φ_1 is an *iwip* automorphism. It induces a train-track map f_1 on the rose R_3; indeed, one checks by hand that ${\varphi_1}^j(a)$ contains no cancellation for every nonnegative integer j since a cancellation can only occur if the word ab occurs by iterating φ_1 on a letter, which is never the case.

The transition matrix $\mathbf{M}_{f_1}$ is primitive and its characteristic polynomial is equal to X^3-X-1. Hence, the dilation coefficient is the dominant eigenvalue $\lambda_{\varphi_1} \approx 1,32$, that is, the smallest Pisot number.

Example 2. Let φ_2 be the automorphism of $F_3 =< a, b, c >$ given by $a \mapsto c$, $b \mapsto c^{-1}a$, $c \mapsto bc^{-1}$. It is the inverse of the substitution $\sigma_2 : a \mapsto ab,\ b \mapsto ca,\ c \mapsto a$ (sometimes called the *flipped Tribonacci substitution*).

The characteristic polynomial of the abelianized matrix A_{φ_2} is $X^3 + X^2 + X - 1$ which is irreducible. Hence, φ_2 is an *iwip* automorphism. One checks that it induces a train-track map f_2 on the rose R_3.

The transition matrix M_{φ_2} is primitive, and its characteristic polynomial $X^3 - X^2 - X - 1$ is irreducible. Hence, the dilation coefficient $\lambda_{\varphi_2} \approx 1.84$ is a Pisot number, the so-called *Tribonacci number*, according to [43]. See also Section 4.2.

Primitive invertible substitutions and iwip *automorphisms.* Let us relate primitive invertible substitutions to *iwip* automorphisms.

Proposition 1.8. *Let σ be an invertible substitution.*

(1) If the characteristic polynomial of the transition matrix $\mathbf{M}_\sigma$ is irreducible over $\mathbb{Z}$, then σ extends to an iwip *automorphism.*

(2) If σ extends to an iwip *automorphism, then σ is primitive.*

Proof. (1) is a consequence of Proposition 1.7, since the abelianized matrix of a substitution is equal to its transition matrix.

To prove (2), suppose that σ is not primitive, and let $f : R_N \to R_N$ be the obvious representation of σ on the rose R_N, with transition matrix $\mathbf{M}_\sigma = \mathbf{M}_f$. By hypothesis, the transition matrix $\mathbf{M}_f$ is not primitive. Then, according to Remark 1, there exists some $K > 0$ such that $\mathbf{M}_f^K$ is not irreducible; so there exist two edges e, e' of the rose R_N such that for all $n \in \mathbb{N}$, e' does not occur as a subpath of $f^{Kn}(e)$. Denote by R_e the minimal f^K-invariant subgraph of R_N containing e. Then R_e is a nontrivial subgraph of R_N, since it does not contain e'. The subgraph R_e defines a proper σ^k-invariant free factor F of F_N, and thus σ cannot represent an *iwip* automorphism. □

Note that the primitivity of an invertible substitution does not imply that this substitution extends to an *iwip* automorphism. A simple counter-example is given by the invertible substitution $\sigma : a \mapsto bac,\ b \mapsto ba,\ c \mapsto ca$. Indeed, σ is clearly primitive, but fixes bc^{-1} and so fixes the free factor $< bc^{-1} >$ it generates.

Let us mention the following more elaborate example we have learnt from M. Lustig. Consider the invertible substitution $\sigma : a \mapsto abcdea,\ b \mapsto abcde,\ c \mapsto ac,\ d \mapsto ad,\ e \mapsto ea$. Then σ is primitive, but one can check that F_5 splits as a σ-invariant free product $< a, b, c, c^{-1}de > * < c^{-1}d >$. It is an interesting question to find a simple characterization of the primitive substitutions that are *iwip* automorphisms.

2. Attractive symbolic lamination for an *iwip* automorphism

The aim of this section is to introduce the attractive symbolic lamination associated with a primitive train-track map. Let us stress the fact that this attractive lamination is highly non-canonical. Indeed, the outer automorphism Φ is given by the images of a given basis of F_N by one of the representatives φ of Φ, from which we derive, using the algorithm of Bestvina-Handel (Theorem 1.5), a train-track map representing Φ. We thus can build the attractive symbolic lamination associated with f.

At the end of the paper, in the Appendix, we will define the attractive algebraic lamination associated with the outer automorphism Φ, and prove that the symbolic lamination is a representative of this algebraic lamination. Hence, this shows that it has intrinsic properties that only depend on the outer automorphism Φ.

We will however only need the symbolic lamination in the rest of the paper; the interest of the algebraic lamination here is only to prove that this construction has some intrinsic properties. Let us note that there are interesting related questions: the shift gives a natural dynamical system on the symbolic lamination. It seems very probable (see, e.g., [11]) that there is a related flow on the corresponding algebraic lamination associated with an outer automorphism, but the existence and definition of this flow remain unclear for us.

We now consider an *iwip* outer automorphism Φ, and we choose a primitive train-track representative map $f : G \to G$. By definition, when we iterate f on any edge e, we obtain a path in the graph (there is no cancellation).

Using the train-track property, we can associate with f a substitution σ_f, by considering edges e and e^{-1} as different letters. This is possible because no cancellation occurs when we iterate f on any edge. We denote by $\mathcal{A} = \{e_1, e_1^{-1}, \ldots, e_k, e_k^{-1}\}$ the set of names of oriented edges of G (By abuse of notation, we make no distinction between the topological space that is the edge e, and its name $e \in \mathcal{A}$).

Definition 2.1. Let $f : G \to G$ be a train-track map. Let $\mathcal{A} := \{e_1, e_1^{-1}, \ldots, e_k, e_k^{-1}\}$ be the $2k$-letter alphabet of names of oriented edges. We denote by h the natural map from the set of paths in G to $\mathcal{A}^*$ which, with any finite path, associates its *name*, that is, the word given by the names of successive edges contained in the path. We define the *double substitution* σ_f associated with f as the substitution that maps every name of an oriented edge to the names of the successive oriented edges contained in $f(e)$, that is:

$$\forall e \in \mathcal{A},\ \sigma_f(e) = h(f(e)).$$

Example 3. The automorphism $\varphi_1 :\ a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b$ was introduced in Example 1. Its double substitution is the 6-letter substitution $\sigma_{\varphi_1} : a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b,\ a^{-1} \mapsto c^{-1},\ b^{-1} \mapsto a^{-1}c,\ c^{-1} \mapsto b^{-1}$. Note that the incidence matrix of σ_{φ_1} is a primitive matrix. Indeed, $\sigma_{\varphi_1}^{11}(a)$ contains all the letters and the image of any letter eventually contains a.

The double substitution σ_f admits a remarkable symmetry, induced by the inverse map $e \mapsto e^{-1}$ and its extension to finite words.

Definition 2.2. The *flip map* Θ on $\mathcal{A}^*$ is defined by $\Theta(w_1 w_2 \ldots w_n) = w_n^{-1} \ldots w_2^{-1} w_1^{-1}$.

By abuse of language, we extend the definition of the flip map Θ to $\mathcal{A}^{\mathbb{Z}}$ as follows: if $x = (x_i) \in \mathcal{A}^{\mathbb{Z}}$, then $\Theta(x) = (y_i) \in \mathcal{A}^{\mathbb{Z}}$ with $y_i = x_{-i-1}^{-1}$ for all $i \in \mathbb{Z}$.

Proposition 2.3. *The double substitution σ_f commutes with the flip.*

The proof is immediate from the definition of f, because the statement holds for letters. We can now define the orientability of the substitution σ_f:

Definition 2.4. We say that the double substitution σ_f defined above is *orientable* if there exists a subset $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{E}$ and $\Theta(\mathcal{E})$ form a partition of $\mathcal{A}$, and σ_f preserves $\mathcal{E}^*$ ($\sigma_f(\mathcal{E}^*) \subset \mathcal{E}^*$) or exchanges $\mathcal{E}^*$ and $\Theta(\mathcal{E}^*)$ ($\sigma_f(\mathcal{E}^*) \subset \Theta(\mathcal{E}^*)$). Otherwise, we say that σ_f is *nonorientable.*

If σ_f is orientable, we say that σ_f is *orientation preserving* if it preserves $\mathcal{E}^*$, and *orientation reversing* if it exchanges $\mathcal{E}^*$ and $\Theta(\mathcal{E})^*$. In this last case, σ_f^2 is orientation preserving.

Remark 5. The geometrical meaning is clear: if σ_f is orientation preserving, it means that we can give a global orientation to the topological graph G such that f preserves this orientation; the image of any positively oriented (for this orientation) edge only contains positively oriented edges. Of course, the opposite orientation is also preserved by f. The combinatorial meaning is also clear: if σ_f is orientation preserving, then up to a replacement of some edges by their inverses, f is in fact a substitution.

Proposition 2.5. *The double substitution σ_f associated with a primitive train-track map f is primitive if σ_f is nonorientable. If σ_f is orientable and orientation preserving, it splits in two primitive components which are exchanged by the flip Θ. If σ_f is orientation-reversing, it exchanges the two primitive components associated with its square.*

Proof. Suppose, first, that σ_f is orientable and orientation preserving, and let $\mathcal{E}$ be the associated subset of $\mathcal{A}$. Let e be any element of $\mathcal{E}$, and let e' be any edge. By primitivity of the train-track map f, we know that there is an integer n such that $f^n(e)$ contains e' or e'^{-1}; but exactly one of these two letters is in $\mathcal{E}$. This implies that the restriction of the action of σ_f on $\mathcal{E}$ is primitive, and the same is true by symmetry for $\Theta(\mathcal{E})$. A similar proof works in the orientation reversing case.

Suppose now that σ_f is nonorientable. By primitivity, there exists n such that, for any pair e, e' of edges, either e' or e'^{-1} occurs in $f^n(e)$. Consider, on the set $\mathcal{A}$ of oriented edges, the relation "e' occurs in $f^n(e)$", and let B be a strongly connected component for the transitive closure of this relation. It is clear that $\Theta(B)$ is also a strongly connected component. We can also consider the set of nonoriented edges (pairs $(e, \Theta(e))$), and define the same relation. It is clear that B projects to a strongly connected component for this relation. But primitivity implies that the only such component is the complete set; hence B projects to the complete set. If it is disjoint of $\Theta(B)$, then σ_f is orientable which contradicts the hypothesis. Hence, it is all of $\mathcal{A}$, and σ_f is primitive. □

Example 4. Consider, for instance, the train-track map $\varphi : a \mapsto c,\ b \mapsto a^{-1},\ c \mapsto ab^{-1}$. One checks that $\mathbf{M}_{\sigma_\varphi}$ is not primitive. Indeed, σ_φ is orientable, since it stabilizes the letters $\{a, b^{-1}, c\}$. Renaming $b' = b^{-1}$, we can check immediately that φ is in fact the substitution $\varphi : a \mapsto c,\ b' \mapsto a,\ c \mapsto ab'$.

We are now in a position to define the main topic of our study:

Definition 2.6. Let f be a train-track map, and σ_f be the associated double substitution. If σ_f is nonorientable, we define the *attractive symbolic lamination* associated with f as the set X_{σ_f} of the symbolic dynamical system (X_{σ_f}, S) defined by σ_f, and we denote it by L_f.

If σ_f is orientable, the substitution (taking the square in the orientation-reversing case) splits in two disjoint primitive substitutions conjugated by Θ; we define the attractive symbolic lamination L_f in this case as the union of the two systems associated with the two primitive substitutions.

For more details about L_f, see Appendix 1.

We have the following property:

Proposition 2.7. *The attractive symbolic lamination L_f is invariant by the flip Θ.*

Proof. This is immediate by definition in the orientable case; otherwise, it results from the primitivity of the substitution. □

Remark 6. The elements of the lamination can be interpreted as infinite paths in graph G. Particular elements can be obtained in the following way: take an edge e, and consider an inner point of e which is fixed by a power f^n of f (such a point always exists by primitivity). Taking the square of f^n if necessary, we see that the letter e occurs inside the word corresponding to $f(e)$, and the words $h(f^{jn}(e))$ are factors of each other, and converge to a biinfinite word; any such biinfinite word generates L_f, and corresponds to an infinite pointed path fixed by f. The shift corresponds to a move of the origin on the path.

3. Representation of the attractive lamination of a unit Pisot *iwip* automorphism

We now have gathered all the material required for the definition of the central tile of a train track map representing an *iwip* automorphism. This construction will be performed thanks to the double substitution σ_f (Definition 2.1). We first recall in Section 3.1 the construction of the central tile in the classic case of a substitution. We then specifically consider the case of a unit Pisot *iwip* automorphism in Section 3.2. The topological properties of the central tile are detailed in Section 3.3. The action of a domain exchange acting on it is studied in Section 3.4. We conclude this section with dynamical properties in Section 3.5.

3.1. **Central tile of a Pisot substitution.** As reviewed in the introduction, central tiles are compact sets with a fractal boundary which are attractors of some graph-directed Iterated Function System. They were first introduced by Rauzy [43] to produce geometric representations of substitutive dynamical systems for some discrepancy motivations, while Thurston [51] extended them to numeration systems with a noninteger basis. Both constructions apply quite naturally to the class of irreducible unit Pisot substitutions.

Definition 3.1. A primitive substitution is said to be *unit Pisot* if its dominant eigenvalue β is a unit Pisot number.

A primitive substitution is said to be *irreducible* if the characteristic polynomial of its incidence matrix is irreducible. There exist various extensions of the notion of central tiles to more general situations, namely to the non-unit case [47], to the non-Pisot case [8], or to the reducible case [12, 31, 25]. The construction we describe here is based on the construction of central tiles for reducible unit Pisot substitutions developed in [12, 31]. Note that here we do not suppose that the substitution is *unimodular*, meaning that $\det M_\sigma = \pm 1$. The only assumption that we need is that the dominant eigenvalue is a unit, even if the matrix M_σ is not invertible over $\mathbb{Z}$.

There are several construction methods for central tiles. One first approach inspired by the seminal paper [43] is based on formal power series, and is developed in [38, 39, 18, 19]. A second approach via iterated function systems (IFS) and generalized substitutions has been developed on the basis of ideas from [29] in [9, 44, 28, 49], with special focus on self-similar properties of the Rauzy fractals and relations with discrete planes [6, 7]. Similar sets have also been introduced and studied in the framework of β-numeration by S. Akiyama in [1, 2, 4, 3], inspired by [51]. Both constructions are equivalent to a third one, which involves projecting a stair representing by abelianization the periodic point of a substitution on a contracting subspace to get a bounded set [11, 13]. This last construction is the most natural, hence it is the starting point of our construction.

The discrete line. Let u be a two-sided periodic point of a primitive unit Pisot substitution σ on k letters. Let us embed this infinite word u as a discrete line in $\mathbb{R}^k$ by successively replacing each letter of u by the corresponding vector in the canonical basis, as depicted in Fig. 3.1. We thus

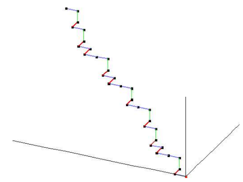

FIGURE 3.1. The discrete line $\mathcal{D}_\sigma$

define the discrete line $\mathcal{D}_\sigma$

$$\mathcal{D}_\sigma := \{\mathbf{l}(u_0 \cdots u_i);\ i \in \mathbb{N}\}. \tag{3.1}$$

Let us recall that $\mathbf{l}$ stands for the abelianization map of the free monoid. By discrete line we mean here a countable set of points which are vertices of a stair, and not really the stair itself.

Algebraic decomposition of the space. We now need to introduce a suitable decomposition of $\mathbb{R}^k$ with respect to eigenspaces of the incidence matrix $\mathbf{M}_\sigma$ associated with the eigenvalue β. We denote by d the algebraic degree of β; one has $d \leqslant k$, since the characteristic polynomial of $\mathbf{M}_\sigma$ may be reducible.

Let $r-1$ stand for the number of real conjugates of β; each corresponding eigenspace has dimension one according to the primitivity assumption and Perron-Frobenius' theorem. Let $2s$ denote the number of complex conjugates of β; each eigenvector pair with its complex conjugate generates a stable 2-dimensional plane. We call *β-contracting space* of the matrix $\mathbf{M}_\sigma$ the subspace $\mathbb{H}_c$ generated by eigenspaces associated with β-conjugates. It has dimension $r+2s-1 = d-1$. Then, by construction, the linear map associated with the matrix $\mathbf{M}_\sigma$ maps $\mathbb{H}_c$ to $\mathbb{H}_c$. We denote by $\mathbf{h}_\sigma :\ \mathbb{H}_c \to \mathbb{H}_c$ its restriction to $\mathbb{H}_c$; it is a contraction with eigenvalues the conjugates of β.

We denote by $\mathbb{H}_e$ the *β-expanding line* of $\mathbf{M}_\sigma$, i.e., the real line generated by a β-eigenvector. Let P be the characteristic polynomial of $\mathbf{M}_\sigma$, and Q the minimal polynomial of β; we can write $P = QR$, where R is a polynomial prime with Q. We denote by $\mathbb{H}_r$ the kernel of $R\,(\mathbf{M}_\sigma)$; it is an invariant space for $\mathbf{M}_\sigma$, which is a complement in $\mathbb{R}^n$ to $\mathbb{H}_c \oplus \mathbb{H}_e$. Let $\pi_c : \mathbb{R}^n \to \mathbb{H}_c$ be the projection onto $\mathbb{H}_c$ along $\mathbb{H}_e \oplus \mathbb{H}_r$, according to the natural decomposition $\mathbb{R}^k = \mathbb{H}_c \oplus \mathbb{H}_e \oplus \mathbb{H}_r$. Then, the relation $\mathbf{l}(\sigma(w)) = \mathbf{M}_\sigma \mathbf{l}(w)$, for all $w \in \mathcal{A}^*$ implies the following commutation relation:

$$\forall w \in \mathcal{A}^*,\ \pi_c(\mathbf{l}(\sigma(w))) = \mathbf{h}_\sigma \pi_c(\mathbf{l}(w)).$$

Central tile. In the irreducible case, the Pisot assumption implies that the discrete line of σ remains at a bounded distance from the expanding direction of the incidence matrix. In the reducible case, the discrete line may have other expanding directions, but the projection of the discrete line by π_c still provides a bounded set in $\mathbb{H}_c \simeq \mathbb{R}^{d-1}$.

Definition 3.2. Let σ be a primitive unit Pisot substitution with dominant eigenvalue β. The *central tile* of σ is the projection on the β-contracting plane of the discrete line associated with any periodic point $u = (u_i)_{i\in\mathbb{Z}}$ of σ:

$$\mathcal{T} = \overline{\{\pi_c(\mathbf{l}(u_0 \cdots u_{i-1}));\ i \in \mathbb{N}\}}.$$

Subtiles of the central tile $\mathcal{R}$ are naturally defined, depending on the letter associated with the vertex of the discrete line that is projected. One thus gets for $1 \leqslant j \leqslant k$

$$\mathcal{T}_\sigma(a_j) = \overline{\{\pi\,(\mathbf{l}(u_0 \cdots u_{i-1}))\,;\ i \in \mathbb{N},\ u_i = a_j\}}.$$

By definition, the central tile $\mathcal{T}_\sigma$ consists of the finite union of its subtiles $\mathcal{T}_\sigma(a_j)$. To ensure that the subtiles are disjoint, we recall from [9] the following combinatorial condition on substitutions.

Definition 3.3. A substitution σ over the alphabet $\mathcal{A}$ satisfies the *strong coincidence condition* if for every pair $(b_1, b_2) \in \mathcal{A}^2$, there exists n such that: $\sigma^n(b_1) = p_1 a s_1$ and $\sigma^n(b_2) = p_2 a s_2$ with $\mathbf{l}(p_1) = \mathbf{l}(p_2)$,where $\mathbf{l}$ denotes the abelianization map.

This condition is satisfied by every unimodular irreducible Pisot substitution over a two-letter alphabet [10]. It is conjectured that every substitution of Pisot type satisfies the strong coincidence condition.

Theorem 3.4. *Let σ be a primitive Pisot unit substitution. Let d be the degree of its dominant eigenvalue. The central tile $\mathcal{T}_\sigma$ is a compact set with nonempty interior in $\mathbb{R}^{d-1}$. Hence it has non-zero measure. Each subtile is the closure of its interior. The subtiles of $\mathcal{T}_\sigma$ are solutions of the following graph-directed affine Iterated Function System:*

$$\forall a \in \mathcal{A},\ \mathcal{T}_\sigma(a) = \bigcup_{b \in \mathcal{A}, \sigma(b) = pas} \mathbf{h}_\sigma(\mathcal{T}_\sigma(b)) + \pi_c(\mathbf{l}(p)). \tag{3.2}$$

We assume, furthermore, that σ satisfies the strong coincidence condition. Then the subtiles have disjoint interiors.

Proof. See for instance [43] for the Tribonacci case, [47, 49, 31] for (3.2), [9] for the last statement, and for more details, the survey [13] and [11]. □

Example 5. Let $\sigma_1 :\ a \mapsto ab,\ b \mapsto c,\ c \mapsto a$. This substitution is invertible and its inverse is the automorphism $\varphi_1 : a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b$ introduced in Example 1. The substitution σ_1 is irreducible, primitive, and Pisot unit. Its dominant eigenvalue β_1 is the second smallest Pisot number; it satisfies $\beta_1^3 = \beta_1^2 + 1$. The central tile of σ_1 is shown in Fig. 3.1; ; the largest shaped tile is the tile $\mathcal{T}_{\sigma_3}(a)$, the middle size shape is $\mathcal{T}_{\sigma_3}(b)$ while the smallest shaped tile is $\mathcal{T}_{\sigma_3}(c)$.

Let $\sigma_2 :\ a \mapsto ab,\ b \mapsto ca,\ c \mapsto a$ be the so-called *flipped Tribonacci substitution*. Its inverse $\varphi_2 : a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto bc^{-1}$ was introduced in Example 2. Like the previous example, σ_2 is irreducible, primitive, and unit Pisot. Its dominant eigenvalue satisfies $\beta^3 = \beta^2 + \beta + 1$. Note that the train-track representative of σ_2^{-1} does not have the same dilation coefficient as σ_2. The central tile of σ_2 is shown in Fig. 3.1.

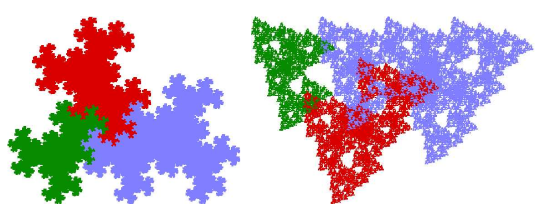

FIGURE 3.2. Central tiles for the substitutions σ_1, σ_2.

Dynamical systems acting on the central tile. Two maps act in a natural way on the symbolic system X_σ, namely the shift map and the substitution.

From the definition of the subtile, one can translate any point of a subtile $\mathcal{T}_\sigma(a)$, for $a \in \mathcal{A}$ by the projection on the contracting plane of the corresponding canonical vector, without leaving the central tile:

$$\begin{aligned} \mathcal{T}_\sigma(a) + \pi_c(\mathbf{l}(a)) &= \overline{\pi_c\left(\{\mathbf{t}(u_0 \cdots u_{i-1}a);\ i \in \mathbb{N},\ u_k = a\}\right)} \\ &\subset \overline{\pi_c\left(\{\mathbf{t}(u_0 \cdots u_{i-1}u_k);\ i \in \mathbb{N}\}\right)} = \mathcal{T}_\sigma. \end{aligned}$$

Consequently, on each subtile, the shift map commutes with the following domain exchange

$$E_\sigma : \begin{array}{ccc} \mathcal{T}_\sigma & \to & \mathcal{T}_\sigma, \\ x \in \mathcal{T}_\sigma(a) & \mapsto & x + \pi_c(\mathbf{l}(a)) \in \mathcal{T}_\sigma. \end{array} \tag{3.3}$$

On the whole central tile, the substitution commutes with the contraction $\mathbf{h}_\sigma : \mathbb{H}_c \to \mathbb{H}_c$, defined in the algebraic decomposition and used in Theorem 3.4. Once the strong coincidence condition is satisfied, the subtiles are disjoint almost everywhere, so both actions can be defined on the whole central tile.

It is natural to code, using the partition defined by the subtiles, the action of the domain exchange E_σ over the central tile $\mathcal{T}_\sigma$. According to [43, 9] there exists a coding map from $\mathcal{T}_\sigma$ into the full shift $\mathcal{A}^{\mathbb{Z}}$ that is one-to-one almost everywhere. Moreover, this coding map is onto the substitutive system X_σ.

Theorem 3.5. *Let σ be a primitive Pisot unit substitution that satisfies the coincidence condition. The substitutive dynamical system (X_σ, S) associated with σ is semi-topologically conjugate to the domain exchange E_σ defined on the central tile $\mathcal{T}$ (see (3.3)).*

Proof. See for instance in [9, 19, 12]. See also for more details, the surveys [13, 41]. □

Remark 7. The minimality of the shift map has an interesting consequence: consider any two-sided sequence w in the symbolic dynamical system X_σ and build a discrete line in $\mathbb{R}^d$ as in (3.1). Then the closure of the projection of the vertices of the discrete line describing w on $\mathbb{H}_c$ is exactly the central tile. Similarly, if one projects the negative part of the discrete instead of the positive part, then the resulting set is again the central tile.

3.2. **Central tile of a unit Pisot *iwip* automorphism.** To associate a central tile with a train-track map f representing an *iwip* automorphism, we use the double substitution σ_f defined in Definition 2.1. In all that follows, we focus on the case of nonorientable double substitutions; indeed, the case of an orientable double substitution can be easily reduced to the classic case of a substitution. We study, in this section, the relation between the incidence matrix $\mathbf{M}_{\sigma_f}$ of σ_f and the transition matrix $\mathbf{M}_f$ of f. We then deduce a construction for the central tile.

Let $\varphi \in \mathrm{Aut}(F_N)$ be an *iwip* automorphism, and let Φ be the outer automorphism defined by φ. We consider a train-track map $f : G \to G$ representing Φ. The choice of the train-track map f is now fixed. Note that the notions we introduce in the following may depend on this particular choice. We denote the edges of G by $e_1, e_1^{-1}, \ldots, e_k, e_k^{-1}$. In all that follows, we focus only on the unit Pisot case:

Definition 3.6. An *iwip* automorphism is said to be *unit Pisot* if its dilation coefficient β is a unit Pisot number.

We have seen in Section 1.3 that, while β is defined as the dominant eigenvalue of the primitive transition matrix $\mathbf{M}_f$, it does not depend on the choice of f but only on the outer automorphism Φ.

The transition matrix $\mathbf{M}_f$ provides information on the dilation coefficient of the *iwip* automorphism φ but, as explained in Sec. 2, it is not sufficient for our purpose. The aim of Proposition 3.7 below is to prove in the unit Pisot case that the double substitution σ_f is a primitive substitution (when f is assumed to be nonorientable) whose dominant eigenvalue is a unit Pisot number.

Proposition 3.7. *Let $f : G \to G$ be a train-track map for a unit Pisot outer automorphism Φ. If f is nonorientable, then the double substitution σ_f is a primitive unit Pisot substitution. The dominant eigenvalue of $\mathbf{M}_{\sigma_f}$ is the dilation coefficient of Φ.*

Proof. Since the train-track map is nonorientable, the incidence matrix $\mathbf{M}_{\sigma_f}$ of σ_f is primitive, according to Proposition 2.5. We have to prove that its dominant eigenvalue is the dilation coefficient of Φ, that is, the dominant eigenvalue of the transition matrix $\mathbf{M}_f$ of the train-track map.

We introduce two auxiliary transition matrices which distinguish between the edge e_i and the edge e_i^{-1}: we define $\mathbf{M}_f^+$ (resp. $\mathbf{M}_f^-$) as the k-square matrix with (i,j)-entry the number of times the path $f(e_j)$ crosses the edge e_i (resp. e_i^{-1}). Both matrices $\mathbf{M}_f^+$ and $\mathbf{M}_f^-$ have nonnegative integer entries. The transition matrix $\mathbf{M}_f$ satisfies $\mathbf{M}_f = \mathbf{M}_f^+ + \mathbf{M}_f^-$.

We define the *abelianization matrix* $\mathbf{A}_f$ of f as $\mathbf{A}_f := \mathbf{M}_f^+ - \mathbf{M}_f^-$. Then we have $\mathbf{M}_{\sigma_f} = \begin{bmatrix} \mathbf{M}_f^+ & \mathbf{M}_f^- \\ \mathbf{M}_f^- & \mathbf{M}_f^+ \end{bmatrix}$, so $\mathbf{P}\mathbf{M}_{\sigma_f}\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{M}_f & 0 \\ 0 & \mathbf{A}_f \end{pmatrix}$ with $\mathbf{P} = \begin{pmatrix} \mathbf{I}_k & \mathbf{I}_k \\ \mathbf{I}_k & -\mathbf{I}_k \end{pmatrix}$.

Hence, the eigenvalues of $\mathbf{M}_{\sigma_f}$ are the eigenvalues of $\mathbf{M}_f$ or $\mathbf{A}_f$. To prove the proposition, we just check that the dominant eigenvalue β of $\mathbf{M}_f$ also dominates the eigenvalues of $\mathbf{A}_f$. Consider an eigenvalue μ of $\mathbf{A}_f$ and an associated eigenvector $\mathbf{v} = (v_1, \ldots, v_n)$. Denoting by m_{ij}^+ (resp. m_{ij}^-) the (i,j)-coefficient of $\mathbf{M}_f^+$ (resp. $\mathbf{M}_f^-$), we obtain that for all $i \in \{1, \ldots, n\}$,

$$\sum_{1 \leqslant i \leqslant n} (m_{ij}^+ - m_{ij}^-) v_i = \mu v_i.$$

Hence

$$\begin{array}{rcl} \mu|v_i| & \leqslant & \sum_{1 \leqslant i \leqslant n} |m_{ij}^+ - m_{ij}^-||v_i| \\ & \leqslant & \sum_{1 \leqslant i \leqslant n} (m_{ij}^+ + m_{ij}^-)|v_i|. \end{array}$$

Thus $|\mu|\mathbf{v}_0 \leqslant \mathbf{M}_f \mathbf{v}_0$, where $\mathbf{v}_0 = (|v_1|, \ldots, |v_n|)$, which implies (see for instance [45]) that $|\mu| \leqslant \beta$. $\square$

We are now able to define the central tile for a nonorientable double substitution.

Definition 3.8. Let Φ be a unit Pisot outer automorphism and let $f : G \to G$ be a train-track map representing Φ. If the double substitution σ_f is nonorientable, then the *central tile of f* is the central tile associated with its double substitution σ_f. We denote it by $\mathcal{T}_f$. The subtiles are denoted by $\mathcal{T}_f(e)$.

Theorem 3.9. *Let $f : G \to G$ be a nonorientable train-track for a unit Pisot* iwip *outer automorphism. Let $2k$ be the number of edges of G. Let d be the degree of the dilation coefficient of f. The central tile of f is a compact subset with nonempty interior, hence non-zero measure, of a $(d-1)$-dimensional subspace of $\mathbb{R}^{2k}$. It is divided into $2k$ subtiles.*

Proof. Since f acts on k edges, σ_f is a substitution on $2k$ letters, so the central tile is a subset of $\mathbb{R}^{2k}$ and is divided into $2k$ tiles. Let β denote the dilation coefficient of f. We have proved (Proposition 3.7) that β is also the dominant eigenvalue of $\mathbf{M}_{\sigma_f}$. By construction, the central tile is a subset of the subspace of $\mathbb{R}^{2k}$ generated by the eigenvectors of $\mathbf{M}_{\sigma_f}$ associated with the conjugates of β. Since f is nonorientable, $\mathbf{M}_{\sigma_f}$ is primitive so β is a simple eigenvalue of $\mathbf{M}_{\sigma_f}$. Hence, all the conjugates are also simple eigenvalues of $\mathbf{M}_{\sigma_f}$. Finally, the contracting space of $\mathbf{M}_{\sigma_f}$ has dimension $(d-1)$. $\square$

An alternative construction via discrete lines. The following property of f allows us to construct the central tile directly from the train-track f, i.e., in a k-dimensional space instead of a $2k$-dimensional space.

Lemma 3.10. *Let σ_f be a primitive double substitution (see Definition 2.1) on the $2k$-letter alphabet $\mathcal{A} = \{e_1, \ldots, e_k, e_1^{-1}, \ldots, e_k^{-1}\}$. Then the projection π_c on $\mathbb{H}_c$ maps a letter and its inverse to the same vector:*

$$\forall i \leqslant k, \quad \pi_c(\mathbf{l}(e_i)) = \pi_c(\mathbf{l}(e_i^{-1})).$$

Proof. We recall that $\mathbf{l}(e_i) = \mathbf{e}_i$, and $\mathbf{l}(e_i^{-1}) = \mathbf{e}_{i+k}$, for $1 \leqslant i \leqslant k$, where $(\mathbf{e}_1, \cdots, \mathbf{e}_{2k})$ denotes the canonical basis of $\mathbb{R}^{2k}$. The vector space $\mathbb{H}_1$ spanned by $\mathbf{e}_1 + \mathbf{e}_{k+1}, \mathbf{e}_2 + \mathbf{e}_{k+2}, \ldots, \mathbf{e}_k + \mathbf{e}_{2k}$ is a stable subspace of $\mathbf{M}_{\sigma_f}$, as well as the vector space $\mathbb{H}_2$ spanned by $\mathbf{e}_1 - \mathbf{e}_{k+1}, \mathbf{e}_2 - \mathbf{e}_{k+2}, \ldots, \mathbf{e}_k - \mathbf{e}_{2k}$. The spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ are orthogonal.

Let β_i be a conjugate of the dominating eigenvalue of σ_f and $\mathbf{v}$ a k-dimensional β_i-eigenvector of the transition matrix $\mathbf{M}_f$. Then, as a consequence of the proof of Proposition 3.7, $(\mathbf{v}, \mathbf{v})$ is a $2k$-dimensional β_i-eigenvector of the incidence matrix $\mathbf{M}_{\sigma_f}$. We have also seen that β_i is a simple eigenvalue of M_{σ_f}. Hence, the contracting subspace $\mathbb{H}_c$ of M_{σ_f} is generated by vectors $(\mathbf{v}, \mathbf{v})$ and $\mathbb{H}_c \subset \mathbb{H}_1$. Consequently, $\mathbb{H}_c$ is orthogonal to $\mathbb{H}_2$, so $\pi_c(\mathbb{H}_2) = 0$. Each vector $\mathbf{l}(e_i) - \mathbf{l}(e_i^{-1}) = \mathbf{e}_i - \mathbf{e}_{k+i}$ belongs to $\mathbb{H}_2$, hence $\pi_c(\mathbf{l}(e_i)) = \pi_c(\mathbf{l}(e_i^{-1}))$. $\square$

As a consequence, the central tile of an *iwip* train-track map can be directly constructed by using the transition matrix $\mathbf{M}_f$ of the train-track map f and one two-sided periodic point (see Remark 7): one simply has to represent a letter and its inverse by the same canonical vector.

Corollary 3.11. *Let Φ be a unit Pisot outer automorphism with a nonorientable train-track representative $f : G \to G$. Let $e_1, e_1^{-1}, \ldots e_k, e_k^{-1}$ denote the edges of G. Let $\mathbf{t} : G \to \mathbb{R}^k$ be the map that sends an edge e and its inverse e^{-1} to the same canonical vector: $\mathbf{t}(e_i) = \mathbf{t}(e_i^{-1}) = \mathbf{e}_i \in \mathbb{R}^k$, for $1 \leqslant i \leqslant k$. Let β be the dilation coefficient of f.*

Let $\widetilde{\mathbb{H}_c} \subset \mathbb{R}^k$ denote the β-contracting space of the transition matrix of f and $\widetilde{\pi}_c$ the projection on $\tilde{\mathbb{H}}_c$ along the invariant complementary space. Then the central tile $\mathcal{T}_f$ of f is the closure of the projection by $\widetilde{\pi}_c$ on $\tilde{\mathbb{H}}_c$ of the k-dimensional stair representing a periodic point u of f:

$$\mathcal{T}_f = \overline{\widetilde{\pi}_c(\{\mathbf{t}(u_0) + \cdots + \mathbf{t}(u_i);\ i \in \mathbb{N}\})}.$$

3.3. **Topological properties.** We recall that we denote by $\mathbf{h}_f : \mathbb{H}_c \to \mathbb{H}_c$ the contraction obtained as the restriction of the linear map associated with the matrix $\mathbf{M}_f$ on $\mathbb{H}_c$; in the eigenvector basis $(\mathbf{u}_{\beta^{(2)}}, \cdots, \mathbf{u}_{\beta^{(d)}})$, $\mathbf{h}_f$ multiplies the coordinate of index i by $\beta^{(i)}$, for $2 \leqslant i \leqslant d$.

Proposition 3.12. *Let $f : G \to G$ be an nonorientable train-track for a unit Pisot* iwip *outer automorphism. Let $2k$ be the number of edges of G and let d be the degree of the dilation coefficient. Then*

1. *each subtile is the closure of its interior;*
2. *the subtiles of $\mathcal{T}_f$ are solutions of the following graph-directed self-affine Iterated Function System:*

$$\forall e \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\},\ \mathcal{T}_f(e) = \bigcup_{\substack{b \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\}, \\ p, s \text{ reduced words}, \\ f(b) = pes}} \mathbf{h}_f(\mathcal{T}_f(b)) + \pi_c(\mathbf{l}(p)). \tag{3.4}$$

3. *They satisfy the following symmetry properties:*

$$\forall\, e \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\},\ \mathcal{T}_f(e) = -\mathcal{T}_f(e^{-1}) - \pi_c(\mathbf{l}(e)). \tag{3.5}$$

4. *We assume, furthermore, that σ_f satisfies the strong coincidence condition. Then the subtiles have disjoint interiors.*

Proof. We deduce all the statements of Theorem 3.12 from Theorem 3.4, except the symmetry conditions (3.5) that we now prove. We recall that the solutions of the graph-directed self-affine Iterated Function System (3.4) are uniquely determined since β is a Pisot number, according to

[37]. Let us prove that $-\mathcal{T}_f(e^{-1}) - \pi_c(\mathbf{l}(e))$ satisfies (3.4), for $e \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\}$. One has

$$\mathcal{T}_f(e^{-1}) = \bigcup_{\substack{b \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\} \\ f(b) = pes}} \mathbf{h}_f(\mathcal{T}_f(b^{-1})) + \pi_c(\mathbf{l}(s^{-1})).$$

Let us apply $\pi_c \circ \mathbf{l}$ to $f(b) = pes$. From $\pi_c \circ \mathbf{l}(s) = \pi_c \circ \mathbf{l}(s^{-1})$ and $\mathbf{h}_f \pi_c \mathbf{l}(b) = \pi_c \mathbf{l}(f(b))$, one deduces that $\mathbf{h}_f \pi_c \mathbf{l}(b) = \pi_c \mathbf{l}(p) + \pi_c \mathbf{l}(e) + \pi_c \mathbf{l}(s)$, and

$$\begin{aligned}
\left[-\mathcal{T}_f(e^{-1}) - \pi_c \mathbf{l}(e)\right] &= \bigcup_{\substack{b \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\} \\ f(b) = pes}} -\mathbf{h}_f \mathcal{T}_f(b^{-1}) - \pi_c \mathbf{l}(s) - \pi_c \mathbf{l}(e) \\
&= \bigcup_{\substack{b \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\} \\ f(b) = pes}} -\mathbf{h}_f \mathcal{T}_f(b^{-1}) - \mathbf{h}_f \pi_c \mathbf{l}(b) + \pi_c \mathbf{l}(p) \\
&= \bigcup_{\substack{b \in \{e_1^{\pm 1}, \cdots, e_k^{\pm 1}\} \\ f(b) = pes}} \mathbf{h}_f \left[-\mathcal{T}_f(b^{-1}) - \pi_c \mathbf{l}(b)\right] + \pi_c \mathbf{l}(p),
\end{aligned}$$

which ends the proof. □

Example 6. The subtiles of the central tile of the group automorphism $\varphi_1 : a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b$ of Example 1 satisfy

$$\begin{cases} \mathcal{T}_f(a) = \mathbf{h}_f \mathcal{T}_f(b) + \pi_c(\mathbf{l}(c^{-1})) \\ \mathcal{T}_f(b) = \mathbf{h}_f \mathcal{T}_f(c) \\ \mathcal{T}_f(c) = \mathbf{h}_f \mathcal{T}_f(a) \cup \left(\mathbf{h}_f \mathcal{T}_f(b^{-1}) + \pi_c(\mathbf{l}(a))\right) \end{cases} \qquad \begin{cases} \mathcal{T}_f(a^{-1}) = \mathbf{h}_f \mathcal{T}_f(b^{-1}) \\ \mathcal{T}_f(b^{-1}) = \mathbf{h}_f \mathcal{T}_f(c^{-1}) \\ \mathcal{T}_f(c^{-1}) = \mathbf{h}_f \mathcal{T}_f(a^{-1}) \cup (\mathbf{h}_f \mathcal{T}_f(b)) \end{cases}$$

$$\begin{cases} \mathcal{T}_f(a) = -\mathcal{T}_f(a^{-1}) - \pi_c(\mathbf{l}(a)) \\ \mathcal{T}_f(b) = -\mathcal{T}_f(b^{-1}) - \pi_c(\mathbf{l}(b)) \\ \mathcal{T}_f(c) = -\mathcal{T}_f(c^{-1}) - \pi_c(\mathbf{l}(c)). \end{cases}$$

3.4. **Domain exchange.** According to Theorem 3.5, the central tile $\mathcal{T}_f$ of a primitive Pisot substitution can be provided with a domain exchange E_f once the subtiles are disjoint in measure. In the general substitutive case, each subtile $\mathcal{T}_\sigma(a)$ is mapped by the translation vector $\pi_c(\mathbf{l}(a))$ according to (3.3). When the substitution is a double substitution σ_f associated with a train-track map, we have proved (Lemma 3.10) that the translation vectors associated with an edge and its inverse are equal. More precisely, let $e_1, e_1^{-1}, \ldots, e_k, e_k^{-1}$ denote the edges of the train-track map. The central tile is stable under the following action:

$$E_f :\ \mathcal{T}_f \to \mathcal{T}_f,\ x \in \mathcal{T}_f(e) \cup \mathcal{T}_f(e^{-1}) \mapsto x + \pi_c(\mathbf{l}(e)) \in \mathcal{T}_f. \tag{3.6}$$

Corollary 3.13. *The central tile of a unit Pisot* iwip *nonorientable train-track is symmetric with respect to the origin:*

$$\mathcal{T}_f = -\mathcal{T}_f.$$

Proof. From Theorem 3.12, each subtile satisfies $\mathcal{T}_f(e) = -\mathcal{T}_f(e^{-1}) - \pi_c(\mathbf{l}(e))$. Hence $-\mathcal{T}_f(e) = E_f \mathcal{T}_f(e^{-1}) \subset \mathcal{T}_f$. Finally, $-\mathcal{T}_f = -\cup \mathcal{T}_f(e) \subset \mathcal{T}_f$, which also implies the reverse implication. □

Example 7. The central tiles for the group automorphisms φ_1 and φ_2 (Example 1 and 2) of F_3 are 2-dimensional, with 6 subtiles. They are depicted in Fig. 3.3.

It is natural to code, with respect to the partition provided by the $2k$ subtiles, the action of the domain exchange E_f over the central tile $\mathcal{T}_f$. Theorem 3.5 implies that the codings of the orbits of the points in the central tile under the action of the domain exchange E_f are described by the attractive symbolic lamination L_f, that is, the coding map, from $\mathcal{T}_f$ onto the $2k$-letter full shift

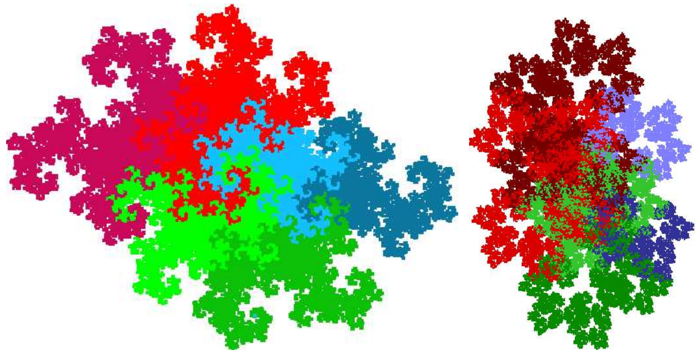

FIGURE 3.3. Central tiles for φ_1, φ_2 (inverse of the flipped Tribonacci substitution).

$\{e_1^{\pm 1}, \dots, e_k^{\pm 1}\}^{\mathbb{Z}}$ is one-to-one almost everywhere, and onto the symbolic dynamical system (L_f, S). We thus have

Theorem 3.14. *Let $f : G \to G$ be a nonorientable train-track for a unit Pisot* iwip *outer automorphism $\varphi \in Out(F_N)$. Let k be the number of edges of G and let d be the degree of the unit Pisot dilation coefficient. We assume, furthermore, that σ_f satisfies the strong coincidence condition. Then the domain exchange E_f (3.6) is defined almost everywhere on the central tile $\mathcal{T}_f$. The attractive symbolic lamination L_f provided with the shift map S is measure-theoretically isomorphic to $(\mathcal{T}_f, E_f)$, i.e., there exists a map $\mu : L_f \to \mathcal{T}_f$ that is is continuous, onto and one-to-one almost everywhere, and that satisfies $\mu \circ S = E_\sigma \circ \mu_\sigma$.*

Example 8. The action of the exchange of domains is illustrated in Fig. 4.4 below for the train-track map φ_2 of Example 2. Pieces move simultaneously by pairs (that consists of a piece and its symmetric for the inverse letter), so there are three translation vectors.

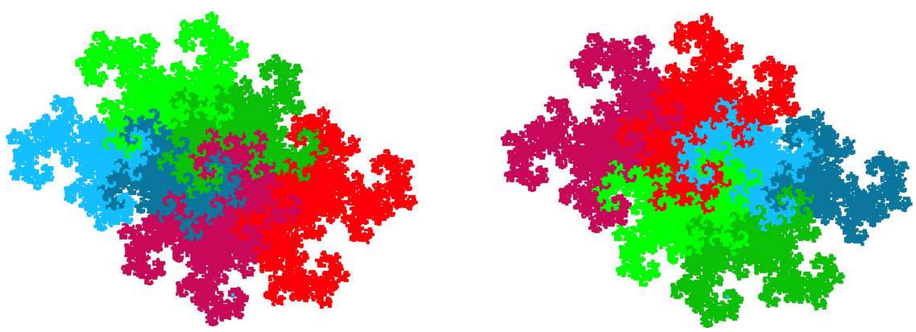

FIGURE 3.4. Exchange of domains for the central tile of φ_2.

3.5. **Dynamical properties.** The domain exchange E_f is defined almost everywhere, but not everywhere, which prevents us from defining a continuous dynamics on the central tile. A solution to this problem consists in "factorizing" the action of E_f on $\mathcal{T}_f$ by the smallest possible lattice so that the translation vectors $\pi_c(\mathbf{l}(e))$ for $e \in \{e_1, \cdots, e_k\}$ do coincide: we thus consider the subgroup

$$\sum_{i=1}^{k-1} \mathbb{Z}\pi_c(\mathbf{l}(e_i) - \mathbf{l}(e_k))$$

of $\mathbb{H}_c$. Let us recall that we mean by irreducible case that the characteristic polynomial of the matrix $\mathbf{M}_f$ is irreducible. In this latter case, this subgroup is discrete, the quotient is a compact group and the domain exchange factorizes into a minimal translation on a compact group.

By a *multiple tiling* of $\mathbb{R}^{d-1}$, we mean according, for instance to [32], arrangements of tiles in $\mathbb{R}^{d-1}$ such that almost all points in $\mathbb{R}^{d-1}$ are covered exactly p times for some positive integer p.

Proposition 3.15. *Let $f : G \to G$ be a nonorientable train-track representative for a unit Pisot* iwip *outer automorphism $\Phi \in Out(F_N)$. Let $2k$ be the number of edges of G and let d be the degree of the unit Pisot dilation coefficient. Let $e_1, e_1^{-1}, \cdots, e_k, e_k^{-1}$ be the edges of graph G. We assume, furthermore, that σ_f satisfies the strong coincidence condition.*

If $d = k$, that is, if the number of edges (up to the orientation) is equal to the algebraic degree of the dilation coefficient, then the central tile generates a lattice multiple tiling of the $(d-1)$-dimensional subspace $\mathbb{H}_c$:

$$\mathbb{H}_c = \mathcal{T}_f + \sum_{i=1}^{k-1} \mathbb{Z}(\pi_c(\mathbf{l}(e_1) - \pi_c(\mathbf{l}(e_k)). \tag{3.7}$$

If this multiple-tiling is a tiling, then (L_f, S) is measure-theoretically isomorphic to a toral translation on the $(d-1)$-dimensional torus $\mathbb{R}^{d-1}$.

In the substitutive case, as soon as the size of the alphabet is larger than or equal to the degree of the expanding eigenvalue, the subgroup $\sum_{i=1}^{k-1} \mathbb{Z}\pi_c(\mathbf{l}(e_i) - \mathbf{l}(e_k))$ has no reason to be discrete. In particular, when σ is a double substitution, the size of the alphabet is at least equal to twice the degree of the expanding eigenvalue. This proposition means that when it is exactly equal to twice the degree, then the symmetries of the substitution imply that the subgroup is discrete, so one gets a multiple tiling.

In the irreducible case, conditions for tiling exist in terms of discrete geometry [31, 11], or of number theory [48, 13]. Most of the conditions (super-coincidences, (F)-property) cannot be directly used in the reducible case. Consequently, we here use the graph condition of [48]; this condition is stated in the irreducible case but it can be generalized to the reducible case, since it only involves the conjugates of the expanding eigenvalue of the substitution. By checking the conditions of [48] on φ_1 and φ_2, we obtain the following result:

Theorem 3.16. *The central tiles for $\varphi_1 : a \mapsto c,\ b \mapsto c^{-1}a,\ c \mapsto b$ and $\varphi_2 : a \mapsto c,\ b \mapsto c^{-1}a$, $c \mapsto bc^{-1}$ generate a periodic tiling of $\mathbb{R}^2$. Hence, the symbolic attractive laminations associated with each train-track maps φ_1 and φ_2 have a pure discrete spectrum. They are both measure-theoretically isomorphic to a translation on the 2-dimensional torus.*

Proof. To prove that one has indeed a tiling, we have to check that $\mathcal{T}$ does not intersect $\mathcal{T} + \pi_c(\mathbf{x})$ for all $\mathbf{x} \in \sum_{i=1}^{k-1} \mathbb{Z}(\mathbf{l}(e_i) - \mathbf{l}(e_k))$. Since we have explicit bounds for the size of $\mathcal{T}$, $\mathcal{T} + \mathbf{x}$ does not intersect $\mathcal{T}$ once the norm of $\mathbf{x}$ is large enough. When it is not the case, following [48], we compute all the algebraic integers of the form $\sum_{i=0}^{j} d_i\beta^{i-j}$, where the sequence of digits (d_i) takes its values in a finite set and satisfies $(\sum_{i\geqslant 0} d_i\beta^i)\mathbf{u}_\beta = \pi_c(\mathbf{x})$ (where $\mathbf{u}_\beta$ is an expanding normalized eigenvector of $\mathbf{M}_{\sigma_f}$). Then we check which sequence of digits (d_i) is a Dumont-Thomas expansion [23]. The final step consists in verifying that the resulting sequences of digits (d_i) correspond to a set of zero-measure in the central tile. All these computations can be done in finite time.

The results of the computation for φ_1 are the following: the central tile $\mathcal{T}$ intersects the six tiles $\mathcal{T}_{\varphi_1} + \mathbf{x}$ with $\mathbf{x} \in \pm\pi_c\{\mathbf{l}(e_3) - \mathbf{l}(e_1), \mathbf{l}(e_2) - \mathbf{l}(e_1), \mathbf{l}(e_2) - \mathbf{l}(e_3)\}$. All the intersections have zero-measure. The central tile of φ_2 intersects height tiles $\mathcal{T}_{\varphi_2} + \mathbf{x}$ with $\mathbf{x} \in \pm\pi_c\{\mathbf{l}(e_3) - \mathbf{l}(e_1), \mathbf{l}(e_2) - \mathbf{l}(e_1), \mathbf{l}(e_2) - \mathbf{l}(e_3), 2\mathbf{l}(e_2) - \mathbf{l}(e_3) - \mathbf{l}(e_1)\}$. □

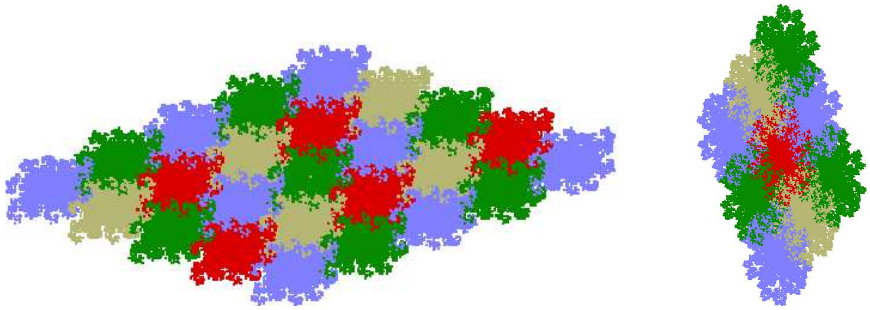

FIGURE 3.5. *Left side*: general aspect of the periodic tiling generated by the train-track $\varphi_1 : a \mapsto c$, $b \mapsto c^{-1}a$, $c \mapsto b$ on F_3. φ_1 is a unit Pisot irreducible nonorientable *iwip* train-track map on the rose R_3 of rank 3. Each tile intersects exactly six neighbourgs; The intersection has zero measure. *Right side*: The central tile of $\varphi_2 : a \mapsto c$, $b \mapsto c^{-1}a$, $c \mapsto bc^{-1}$ also periodically tiles the plane, but each tile has height neighbourgs.

Furthermore, one can even check effectively that the symbolic attractive laminations of φ_1 and φ_2 are not measure-theoretically isomorphic since they have distinct spectra.

4. EXAMPLES

In the previous sections, we introduced two examples φ_1 and φ_2 (Example 1 and 2) corresponding to automorphisms of F_3 that have no cancellation, that is, they directly induce a primitive train-track map on the rose R_3. We give below a few representative examples of more complex situations.

4.1. Example of an *iwip* automorphism on more than 3 letters.

The next example corresponds to an automorphism on 5 letters that is not Pisot.

Example 9. One checks that the automorphism $\varphi_3 : a \mapsto e,\ b \mapsto e^{-1}a,\ c \mapsto b,\ d \mapsto c,\ e \mapsto d$ induces a train-track map f_3 on the rose R_5. Its transition matrix $\mathbf{M}_{f_3}$ has for characteristic polynomial $X^5 - X - 1$, which is irreducible, hence φ_3 is an *iwip* automorphism. Its dilation coefficient is $\lambda_{\varphi_3} \approx 1,16$. However, λ_{φ_3} is not a Pisot number, so that we cannot build a central tile for φ_3.

Note that φ_3 is the inverse of the smallest Pisot number β-substitution $\sigma_3 : a \mapsto ab$, $b \mapsto c$, $c \mapsto d$, $d \mapsto e$, $e \mapsto a$. The characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, so that this substitution is reducible and its dominating eigenvalue is the smallest Pisot number $\beta^3 = \beta + 1$. The central tile of σ_3 is shown in Fig. 4.2.

4.2. Examples of automorphisms that do not induce a train-track map.

The next two examples (Example 10 and 11) correspond to automorphisms that do not directly induce a train-track map. In each case, the transition matrix of the chosen topological representative as a train-track map admits an irreducible characteristic polynomial.

The so-called Tribonacci substitution is defined by $\sigma_4 : a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$. Rauzy's seminal paper [43] is based on a detailed study of this substitution. See also [29, 38, 39, 35]. It is irreducible, primitive, and unit Pisot. Its dominant eigenvalue satisfies $\beta^3 = \beta^2 + \beta + 1$. Its central tile is depicted in Fig. 4.2.

In the following example, we consider the inverse φ_4 of σ_4.

Example 10. The automorphism φ_4 of $F_3 =< a, b, c >$ is given by $a \mapsto c$, $b \mapsto c^{-1}a$, $c \mapsto c^{-1}b$. It does not induce a train-track map on the rose R_3, since a cancellation occurs when computing $\varphi_4^2(c) = \varphi_4(c^{-1}b) = b^{-1}cc^{-1}a = b^{-1}a$.

Thanks to the algorithm of Bestvina-Handel, one obtains a primitive train-track map $f_4 : G_4 \to G_4$ representing φ_4, and to check that φ_4 is *iwip*. The graph G_4 is depicted in Fig. 4.1. It has 2 vertices v_1 and v_2, and 4 edges: A from v_1 to v_1, B and C from v_2 to v_1, and D from v_1 to v_2. The map f_3 is defined by $A \mapsto DC$, $B \mapsto D^{-1}A$, $C \mapsto B$, $D \mapsto C^{-1}$. The correspondence between R_3 and G_4 is given by $a \sim A$, $b \sim DB$, $c \sim DC$.

The transition matrix $\mathbf{M}_{f_4}$ is primitive; its characteristic polynomial is $X^4 - 2X - 1$, and the dilation coefficient is $\lambda_{\varphi_4} \approx 1.39$. Its conjugates are -0.47462, - 0.46035 + 1.13931i, and - 0.46035 - 1.13931 i. Hence, φ_4 is not Pisot and there is no associated central tile.

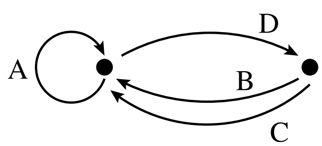

Figure 4.1. A topological representation of φ_4 (the inverse of the Tribonacci substitution) given by the graph G_4 provided with the map $A \mapsto DC$, $B \mapsto D^{-1}A$, $C \mapsto B$, $D \mapsto C^{-1}$.

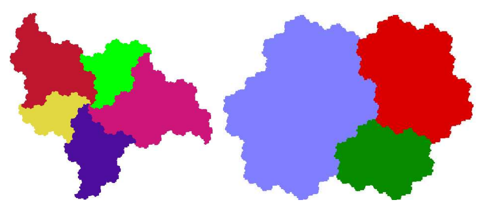

Figure 4.2. Central tile for the β-substitution associated with the smallest Pisot number σ_3 and the Tribonacci substitution σ_4. Their inverse automorphisms are not Pisot and cannot be represented by a central tile.

Example 11. The automorphism φ_5 of $F_3 =< a, b, c >$ given by $a \mapsto c^{-1}bcab^{-1}$, $b \mapsto a^{-1}c^{-1}b^{-1}c$, $c \mapsto a^{-1}c^{-1}b$ does not induce a train-track map on the rose R_3, since a cancellation occurs when computing $\varphi_5(c^{-1}b)$, which occurs in $\varphi_5(a)$.

The algorithm of Bestvina-Handel allows one to obtain a primitive train-track map $f_5 : G_5 \to G_5$, with $G_5 = G_4$, i.e., , the graph on which f_4 (Example 10) is defined (see Fig 4.1), and to check that φ_5 is an *iwip* automorphism. Again, the correspondence between R_3 and G_5 is given by $a \sim A$, $b \sim DB$, $c \sim DC$. The map f_5 is defined by $A \mapsto C^{-1}BDCAB^{-1}D^{-1}$, $B \mapsto D^{-1}B^{-1}C$, $C \mapsto B$, $D \mapsto A^{-1}C^{-1}$.

The transition matrix $\mathbf{M}_{f_5}$ is primitive; its characteristic polynomial is $X^4 - 2X^3 - 2X^2 + 1$, and its dilation coefficient is $\lambda_{\varphi_5} \approx 2.69$, that is a unit Pisot number.

The group automorphism φ_5 of F_3 has a 3-dimensional central tile with eight subtiles since its dilation coefficient has degree 4 and its train-track map f_5 act on a 4-edge graph. It is depicted in Fig. 4.3.

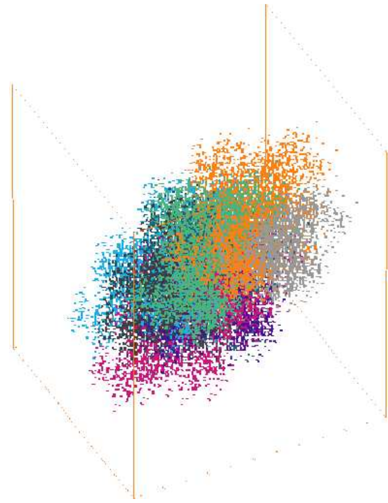

FIGURE 4.3. Central tile for the train-track map f_5 representing φ_5: it is 3-dimensional, with eight subtiles.

4.3. **Example of a train-track map with a reducible matrix.** Example 12 and 13 are examples of automorphisms that induce directly train-track maps, but they have transition matrix with reducible characteristic polynomial.

Example 12. Thanks to the algorithm of Bestvina-Handel, one obtains that the automorphism $\varphi_6 : a \mapsto ab^{-1},\ b \mapsto c,\ c \mapsto d,\ d \mapsto e,\ e \mapsto a$ is an *iwip* automorphism that induces a train-track map f_6 on the rose R_5; indeed, the only possible cancellation can occur within the factors $e^{-1}a$ (or its inverse); one checks that $e^{-1}a$ never occurs since b is always followed by a^{-1}.

Its transition matrix $\mathbf{M}_{f_6}$ has for characteristic polynomial $X^5 - X^4 - 1 = (X^3 - X - 1)(X^2 - X + 1)$, which is reducible. Its dilation coefficient satisfies $X^3 = X + 1$. It is the smallest Pisot number.

This example illustrates that the characteristic polynomial of a primitive matrix is not necessarily irreducible, and that the degree of the dilation coefficient of an *iwip* outer automorphism $\Phi \in \mathrm{Out}(F_N)$ is neither bounded below by the number of edges in a train-track representing Φ, nor even by N.

The central tile of the group automorphism φ_6 of F_5 is 2-dimensional since its dominating eigenvalue has degree 3; it admits 10 subtiles. The central tile is depicted in Fig. 4.2.

The action of the exchange of domains is illustrated in Fig. 4.4 below for the train-track map φ_6 of Example 12. The central tile is divided into 10 subtiles. Pieces move simultaneously by pairs (that consist of a subtile and its symmetric for the inverse letter), so that there are five translation vectors.

4.4. **Example of an orientable automorphism.** We conclude with an example of an orientable automorphism.

Example 13. Consider the automorphism $\varphi_7 : a \mapsto ab^{-1},\ b \mapsto c^{-1},\ c \mapsto d,\ d \mapsto e,\ e \mapsto a$.

Its transition matrix $\mathbf{M}_{f_7}$ has for characteristic polynomial $(X^3 - X - 1)(X^2 - X + 1)$, which is reducible.

Note that by replacing b with b^{-1}, φ_7 becomes the substitution associated with the smallest Pisot number β-substitution σ_3. Hence, φ_7 is an orientable automorphism. The associated attractive lamination corresponds to a substitutive symbolic dynamical system and can be represented by the 2-dimensional central tile shown in Fig. 4.2 with 5 subtiles and no symmetry.

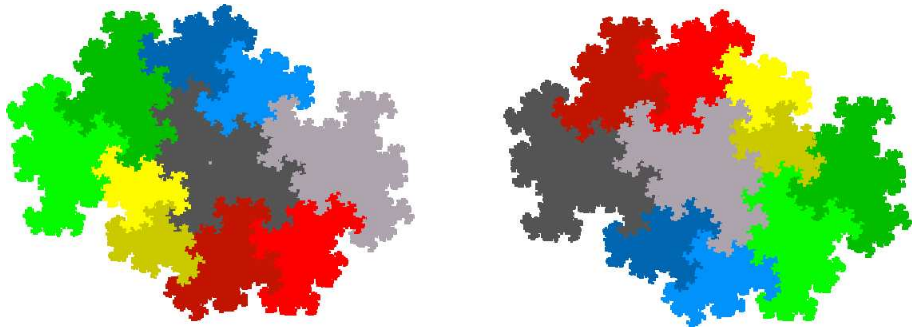

FIGURE 4.4. Exchange of domains for the central tile φ_6. It has 10 subtiles.

5. Perspectives: stability of the representation under conjugation

Our purpose in this paper was to exhibit a connection, in some specific cases, between the geometric representation of symbolic dynamical systems and the theory of free group automorphisms. We plan to explore, in the near future, several natural questions concerning the influence of the choice of the train-track representative on the representation. For instance, does the spectrum of the dynamical system associated with the symbolic lamination depend on the choice of the basis? Are the topological properties of the central tile conserved inside the class of an outer automorphism? Is the nonorientability property satisfied by all train-track representatives of an outer automorphism? More generally, can one define a flow associated with the algebraic attractive lamination, so that the central tile associated with a particular choice of coordinates appears as a section for this flow? Note that, while the present work is restricted to the Pisot case, recent progresses [8] open possibilities in the general hyperbolic case; in that framework, the flow should be replaced by an $\mathbb{R}^k$-action, with k the dimension of the expanding space.

As an illustration, let us consider the two following examples. The substitution $\tau_4 : A \mapsto ACB$, $B \mapsto C$, $C \mapsto A$ is a conjugate of the Rauzy (or Tribonacci) substitution $\sigma_4 : a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$, under the substitution $\mu : A \mapsto ab$, $B \mapsto c$, $C \mapsto a$. Both central tiles of these substitutions are depicted in Fig. 5.1 (left side). One checks that they have similar local structure. A similar phenomenon can be observed for the substitutions $\tau_2 : A \mapsto ABC$, $B \mapsto C$, $C \mapsto A$ and $\sigma_2 : a \mapsto ab$, $b \mapsto ca$, $c \mapsto a$, that also are conjugated by μ. Their central tiles are depicted in Fig. 5.1 (right side); they also seem to present important similarities.

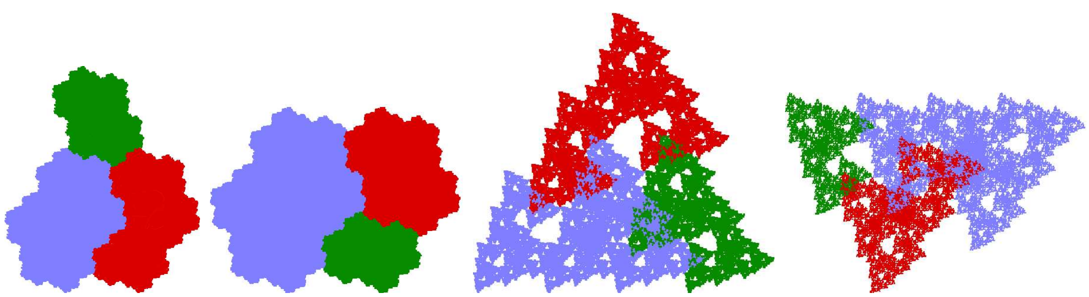

FIGURE 5.1. Central tiles for two pairs (τ_4, σ_4), and (τ_2, σ_2) of Pisot unit irreducible substitutions. Each pair consists of substitutions that belong to the same class of automorphism of F_3.

This similarity can easily be completely explained, as was pointed out to us by Marcy Barge. Indeed, if we define the substitution $\psi_4 : a \mapsto A$, $b \mapsto CB$, $c \mapsto C$, we can check immediately that we have $\sigma_4 = \mu \circ \psi_4$ and $\tau_4 = \psi_4 \circ \mu$ (the use of two different alphabets, upper and lower case, should only be seen as a convenience to keep track of the direction of the conjugacy). This easily implies the commutation relation $\tau_4 \circ \psi_4 = \psi_4 \circ \sigma_4$, which shows that the fixed point v of τ_4 is the image of the fixed point u of σ_4 by ψ_4.

Hence the infinite word v can be coded by the three words A, CB and C, and renaming these three words respectively a, b, c, we recover the infinite word u. Consider now the central tile $\mathcal{T}_{\sigma_4}$ of σ_4, with its natural decomposition $\mathcal{T}_{\sigma_4} = \mathcal{T}_{\sigma_4}(a) \cup \mathcal{T}_{\sigma_4}(b) \cup \mathcal{T}_{\sigma_4}(c)$, and the central tile $\mathcal{T}_{\tau_4} = \mathcal{T}_{\tau_4}(A) \cup \mathcal{T}_{\tau_4}(B) \cup \mathcal{T}_{\sigma_4}(C)$, seen as the closure of the projection of the respective fixed points; from this recoding, we can obtain $\mathcal{T}_{\tau_4}(A)$ as $\mathcal{T}_{\sigma_4}(a)$, $\mathcal{T}_{\tau_4}(B)$ as a translate of $\mathcal{T}_{\sigma_4}(b)$, and $\mathcal{T}_{\tau_4}(C)$ as the disjoint union $\mathcal{T}_{\sigma_4}(b) \cup \mathcal{T}_{\sigma_4}(c)$. This decomposition can be seen in Fig. 5.2, on the left. Let us note that the disjoint unions here are considered up to sets of zero measure.

On the other hand, we have also the commutation relation $\mu \circ \tau_4 = \sigma_4 \circ \mu$, which proves that the fixed point u can be coded by the three words ab, a, and c; from this decomposition, we can obtain $\mathcal{T}_{\sigma_4}(a)$ as the disjoint union $\mathcal{T}_{\tau_4}(A) \cup \mathcal{T}_{\tau_4}(C)$, $\mathcal{T}_{\sigma_4}(b)$ as a translate of $\mathcal{T}_{\tau_4}(A)$, and $\mathcal{T}_{\sigma_4}(c)$ as $\mathcal{T}_{\tau_4}(B)$, see Fig. 5.2 on the right. More precisely, if $\mathbf{M}_\mu$ and $\mathbf{M}_{\psi_4}$ stand for the incidence matrix of the conjugacy substitutions μ and ψ_4, and π_{τ_4} and π_{σ_4} stand for the projections on the β-contracting planes of τ_4 and σ_4, we get the following relations:

$$\left\{\begin{array}{lcl} \mathcal{T}_{\tau_4}(A) & = & \mathbf{M}_{\psi_4}\mathcal{T}_{\sigma_4}(a) \\ \mathcal{T}_{\tau_4}(B) & = & \mathbf{M}_{\psi_4}\mathcal{T}_{\sigma_4}(b) + \pi_{\tau_1}\mathbf{l}(B) \\ \mathcal{T}_{\tau_4}(B) & = & \mathbf{M}_{\psi_4}\mathcal{T}_{\sigma_4}(b) \cup \mathbf{M}_{\psi_4}\mathcal{T}_{\sigma_4}(c) \end{array}\right. \qquad \left\{\begin{array}{lcl} \mathcal{T}_{\sigma_4}(a) & = & \mathbf{M}_{\mu}\mathcal{T}_{\tau_4}(A) \cup \mathbf{M}_{\mu}\mathcal{T}_{\tau_4}(C) \\ \mathcal{T}_{\sigma_4}(b) & = & \mathbf{M}_{\mu}\mathcal{T}_{\tau_4}(A) + \pi_{\sigma_4}\mathbf{l}(a) \\ \mathcal{T}_{\sigma_4}(c) & = & \mathbf{M}_{\mu}\mathcal{T}_{\tau_4}(B). \end{array}\right.$$

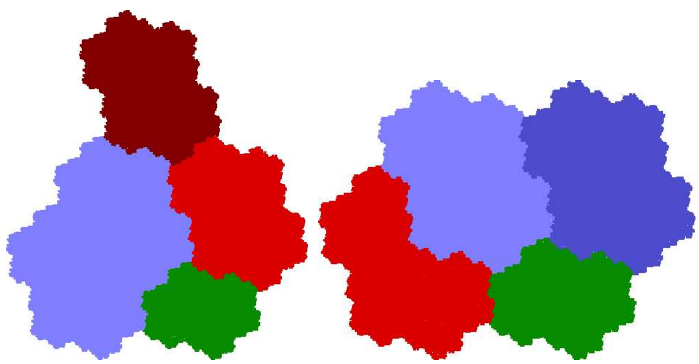

FIGURE 5.2. Relations between central tiles for conjugate substitutions. Left side: the basic subtiles of $\mathcal{T}_{\tau_4}$ (central tile of τ_4) can be rebuilt from the basic subtiles of $\mathcal{T}_{\sigma_4}$ (central tile of Tribonacci substitution σ_4, conjugate to τ_4). Right side: the basic subtiles of $\mathcal{T}_{\sigma_4}$ can be rebuilt from the basic subtiles of $\mathcal{T}_{\tau!la_4}$.

It might seem contradictory to obtain each central tile as a subset of the other; the solution of this apparent paradox is that there is no canonical normalization for the central tile. Using one substitution, we obtain in fact an infinite sequence $(\mathcal{T}_{\sigma_4}^{(n)})$ of decreasing homothetic sets, with relation given by the substitution: if we consider the central tile as the closure of the projection of the discrete line, as explained in subsection 3.1, the recoding, using the substitution, selects a subset of the discrete line which projects to a homothetic set, and this can be iterated ; the other substitution gives another decreasing infinite sequence $(\mathcal{T}_{\tau_4}^{(n)})$ inserted in the first, with $\mathcal{T}_{\sigma_4}^{(n+1)} \subset \mathcal{T}_{\tau_4}^{(n)} \subset \mathcal{T}_{\sigma_4}^{(n)}$, for all n. The best picture in this case is to use the canonical flow associated with the substitution, as defined in [11]. The central tile appears as a section for this flow; the action of the substitution gives in fact an infinite family of homothetic sections. The conjugacy allows us to define another family of sections, which can be inserted in the first one.

The same property is true for the second example, using substitution μ and $\psi_2 : a \mapsto A$, $b \mapsto BC$, $c \mapsto C$; one checks that $\sigma_2 = \mu \circ \psi_2$ and $\tau_2 = \psi_2 \circ \mu$; we leave to the reader the computation of the related decompositions of the central tiles.

Appendix 1: Boundary of a free group and algebraic laminations

We now define intrinsic objects corresponding to what we defined above; the aim of this appendix is to prove that the symbolic lamination defined above as the dynamical system generated by the double substitution is indeed an algebraic lamination in the sense of free groups.

Boundary of a free group. Since the free group F_N is a hyperbolic group, its *Gromov boundary* exists and is denoted by ∂F_N [27, 26, 21]. This boundary appears to be the space of ends of F_N: it is a Cantor set which compactifies F_N. If a basis of F_N is fixed, ∂F_N can be defined as the set of one-sided infinite reduced sequences in this basis, provided with the topology given by the cylinders (one can prove that this topology does not depend on the choice of a basis).

Universal cover of a marked graph. Let G be a marked graph with marking $\tau : R_N \to G$. Let $\tilde{G}$ stand for the universal cover of G, and $pr : \tilde{G} \to G$ for the natural projection; $\tilde{G}$ is a tree, whose vertices have finite valence. There is a natural action of $F_N = \pi_1(G, *)$ on $\tilde{G}$ by deck transformations. Since $\tilde{G}$ is a simplicial tree whose vertices have finite valence, its (Gromov) boundary $\partial\tilde{G}$ exists and is a Cantor set which compactifies $\tilde{G}$ [26, 21]. The boundary $\partial\tilde{G}$ is naturally identified with ∂F_N by fixing a base point in $\tilde{G}$ and using the marking $\tau : R_N \to G$ (or equivalently the action of F_N on $\tilde{G}$).

An alternative definition is as follows: the boundary $\partial\tilde{G}$ can be defined as the set of equivalence classes of rays in $\tilde{G}$, where two rays (one-sided paths) of $\tilde{G}$ are said to be *equivalent* if their intersection is a ray. Consequently, a line (two-sided path) in $\tilde{G}$ defines two distinct points of $\partial\tilde{G}$ (equivalent classes of rays obtained by cutting the line at a point), thus two distinct points of ∂F_N. Conversely, two distinct points of ∂F_N define a unique line in $\tilde{G}$.

A special case is that of the Cayley graph associated with a basis of the free group, which can be seen as a cover of the rose; one-sided infinite reduced sequences in this basis can be seen as infinite reduced paths (rays) in the Cayley graph.

Action of F_N on F_N and ∂F_N. The group F_N acts on itself by conjugacy. This action is continuous, and one can prove that it extends continuously to an action of F_N on ∂F_N by left translations. We denote by $\partial i_u : \partial F_N \to \partial F_N$ the left translation by u that is defined by $\partial i_u(Y) = uY$ for all $Y \in \partial F_N$.

Action of an automorphism on the boundary. Given an automorphism, it is natural to ask whether it can be continuously extended to an homeomorphism of the boundary of F_N considered as a set of one-sided sequences. As explained earlier, every inner automorphism extends naturally to ∂F_N. More generally, it is proved in [20] that such an extension exists for every automorphism $\varphi \in \mathrm{Aut}(F_N)$; we denote by $\partial\varphi : \partial F_N \to \partial F_N$ the homeomorphism extending φ. The proof relies on the fact that an automorphism φ of F_N is a quasi-isometry for the word metric associated with some given basis. This implies in particular the following fact: fix a basis of F_N; there exists a constant $C > 0$ (depending on the basis and the automorphism φ) such that for all reduced words u, v such as uv is also reduced, then the cancellation occurring when concatenating the reduced images of u and v by φ is bounded by C. This is the key ingredient for proving the existence of $\partial\varphi$.

Laminations. Let us define some useful material about general laminations in free groups. For more details, see [22].

Consider the set $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$ (where Δ is the diagonal in $\partial F_N \times \partial F_N$) of pairs of distinct points in ∂F_N. Note that the topology of ∂F_N induces a topology on $\partial^2 F_N$. Moreover, the action of F_N on ∂F_N induces a diagonal action of F_N on $\partial^2 F_N$. The set $\partial^2 F_N$ admits a distinguished involution $\vartheta : (X, Y) \mapsto (Y, X)$, called the *flip involution*: it is an F_N-equivariant homeomorphism.

Definition .1. An *algebraic lamination* L of F_N is a closed subset of $\partial^2 F_N$ which is F_N-invariant and flip-invariant.

If we consider, as above, ∂F_N as the boundary $\partial \tilde{G}$ of the universal cover of a graph G, an element of $\partial^2 F_N$ can be seen as a biinfinite line in the tree $\tilde{G}$, joining two distinct points on the boundary. This representation depends of course on the choice of the graph G (or on the choice of a basis, in the case of the Cayley graph), while the definition of $\partial^2 F_N$ is intrinsic.

Action of Out(F_N) on the laminations. The group $\mathrm{Out}(F_N)$ acts naturally on the set of algebraic laminations of F_N, in the following way. Consider $\varphi \in \mathrm{Aut}(F_N)$ and $(X, Y) \in \partial^2 F_N$, and define $\partial^2\varphi(X, Y)$ to be $(\partial\varphi(X), \partial\varphi(Y))$; then $\partial^2\varphi$ is an homeomorphism of $\partial^2 F_N$. Note that if L is an algebraic lamination, then $\partial^2\varphi(L)$ is also an algebraic lamination. Moreover, because of the F_N-invariance of L, $\partial^2\varphi(L)$ only depends of the outer class Φ of φ, and we shall denote it by $\Phi(L)$. If $\Phi(L) = L$, one traditionally says that Φ *stabilizes the lamination L.*

Symbolic lamination. Since a point in $\partial^2 F_N$ defines a unique line in $\tilde{G}$, an algebraic lamination L defines a set of lines in $\tilde{G}$, denoted by $L(\tilde{G})$. The *symbolic lamination in G-coordinates* associated with L, denoted by $L(G)$, is the set of lines in G which can be lifted to lines of $L(\tilde{G})$. The lines of $L(G)$ are called *leaves*.

With such a symbolic lamination $L(G)$, one can associate the set $\mathcal{L}(L(G))$, called the *laminary language of L in G-coordinates*, which consists of all the finite paths in G which occur in some leaf of $L(G)$.

The attractive lamination of an iwip *outer automorphism.* To study the dynamics of a free group automorphism, it is interesting to consider its action on the boundary, and to look for fixed points. Similarly as in the case of surface homeomorphisms, it makes sense to look for a lamination that is invariant by an automorphism, and such lamination always exists for an *iwip* automorphism, as detailed below.

Let Φ be an *iwip* outer automorphism, and let f be a primitive train-track representative $f : G \to G$. We define the set $\mathcal{L}_f^+$ according to the following condition: a path w of G belongs to $\mathcal{L}_f^+$ if, and only if, there exists some edge e of G and some $n \geqslant 1$ such that w is a subpath of $f^n(e)$. The following result is proved in [15] (see also [14]):

Theorem .2 (Bestvina-Feighn-Handel)**.** *Let Φ be an* iwip *outer automorphism Φ, and let $f : G \to G$ be a primitive train-track representative. There exists an algebraic lamination L_Φ^+, called the attractive lamination of Φ, whose laminary language in G-coordinates is $\mathcal{L}_f^+$. Moreover:*

- *this algebraic lamination does not depend on the choice of the train-track f representing Φ,*
- *Φ stabilizes L_Φ^+, that is, $\Phi(L_\Phi^+) = L_\Phi^+$.*

Definition .3. Let Φ be an *iwip* outer automorphism Φ, and let $f : G \to G$ be a primitive train-track representative. The *symbolic attractive lamination in G-coordinates* $L_\Phi^+(G)$ is defined as the set of two-sided sequences in $\partial^2 F_N$ whose language is included in $\mathcal{L}_f^+$.

Let us note that one has $\mathcal{L}(L_\Phi^+(G)) = \mathcal{L}_f^+$.

Symbolic dynamics generated by a symbolic lamination. Consider $f : G \to G$ a primitive train-track map representing an *iwip* automorphism Φ. Proposition .5 below states that the symbolic attractive lamination $\mathcal{L}_f^+(G)$ coincides with the lamination associated with the double substitution σ_f.

Remark 8. The analogy with linear algebra can be enlightening. Consider an endomorphism f of $\mathbb{R}^3$, and suppose that f has an eigenspace which is a plane P. Every choice of a basis of $\mathbb{R}^3$ allows one to describe the plane P thanks to an equation. This equation become simpler when the basis contains a basis of P, since this choice of a basis respects the geometry of f. The attractive lamination of an *iwip* outer automorphism $\Phi \in \mathrm{Out}(F_N)$ is an intrinsic object, lying in $\partial^2 F_N$. For each choice of a marked graph, that is, a choice of coordinates, the lamination can be represented as a subset of lines embedded in the universal cover of G. When G is the graph of a train-track map f representing Φ, the subset of lines becomes "nice" since a line of this lamination is given by iterations of f on an edge, and without cancellation. Roughly, we can also say that the choice of the representation respects the geometry.

Let us recall that the flip map Θ is defined in Definition 2.2.

Lemma .4. *Let f be a primitive train-track map representing an iwip outer automorphism. Then there exists p such that for all edge a, the following decomposition holds: $f^p(a) = w_{-1}aw_1$, with w_1, w_{-1} nonempty words. The following two-sided sequence in $\partial^2 F_N$ belongs to the symbolic attractive lamination in G-coordinates:*

$$l_a = \dots f^{(j-1)p}(w_{-1}) \dots f^p(w_{-1})w_{-1} \cdot aw_1 f^p(w_1) \dots f^{(j-1)p}(w_1) \dots \in L_\Phi^+(G).$$

Let $X_{l_a} = \overline{\{S^j l_a;\ j \in \mathbb{Z}\}}$ be the closed-shift orbit of l_a, where S denotes the shift map on $\partial^2 F_N$. Then

- *(X_{l_a}, S) is a minimal symbolic dynamical system;*
- *the symbolic attractive lamination $L_\Phi^+(G)$ is generated by X_{l_a}:*

$$L_\Phi^+(G) = X_{l_a} \cup \Theta(X_{l_a}).$$

Proof Fix an edge a of G. By primitivity of the transition matrix $\mathbf{M}_f$, then there exists $m > 0$ such that all the entries of $\mathbf{M}_f^m$ are greater than 3. Hence, $f^m(a) = x_{-1}ax_1$ or $f^m(a) = x_{-1}a^{-1}x_1$ for all edge a, with nonempty x_{-1}, x_1 (since a or a^{-1} occurs at least three times in $f^m(a)$). Hence, $p = 2m$ satisfies $f^p(a) = w_{-1}aw_1$ for all edge a, with nonempty w_{-1}, w_1. Then for all $j \geqslant 2$, $f^{jp}(a) = w_{-j}aw_j$ with $w_{-j} = f^{(j-1)p}(w_{-1}) \dots f^p(w_{-1})w_{-1}$ and $w_j = w_1 f^p(w_1) \dots f^{(j-1)p}(w_1)$. Note that no cancellation occurs when iterating f on a because f is a train-track map, hence, we can define l_a as follows:

$$l_a = \dots f^{(j-1)p}(w_{-1}) \dots f^p(w_{-1})w_{-1} \cdot aw_1 f^p(w_1) \dots f^{(j-1)p}(w_1) \dots$$

Since the language of l_a is included in $\mathcal{L}_f^+$, we have $l_a \in L_\Phi^+(G)$.

The primitivity of $\mathbf{M}_f$ implies that the line l_a is quasiperiodic as shown in [15]. Consequently, (X_{l_a}, S) is a minimal symbolic dynamical system (see Section 1.1).

The language of every $l \in X_{l_a}$ is the same as that of l_a, i.e., the set of factors of $\{f^{jp}(a), k \in \mathbb{N}\}$. Since it is contained in $\mathcal{L}_f^+$, we have $X_{l_a} \subset L_\Phi^+(G)$. Every finite factor of any two-sided sequence in $\Theta(X_{l_a})$ is included in an iterate $\Theta(f^p(a)) = f^p(a^{-1})$, so we also have $\Theta(X_{l_a}) \subset L_\Phi^+(G)$.

Let b be another edge in the graph. By the definition of p, either b or b^{-1} occurs in $f^p(a)$. If b occurs in $f^p(a)$, then the language of l_b, that is, the set of factors of $\{f^{jp}(b), k \in \mathbb{N}\}$ is included in the language of l_a, hence, $l_b \in X_{l_a}$. If b occurs in $f^p(a^{-1})$, then $l_b \in X_{\Theta(l_a)} = \Theta(X_{l_a})$. Finally, let $l \in L_\Phi^+(G)$. Every factor $l_{[-j,j]}$ is a factor of $f^{pm_j}(e_j)$. Since the e_j's are finite, there exists b such that the language of l is included in the set of factors of $\{f^{jp}(b), k \in \mathbb{N}\}$. Hence $l \in X_{l_b} \subset X_{l_a} \cup \Theta(X_{l_a})$, which implies $L_\Phi^+(G) = X_{l_a} \cup \Theta(X_{l_a})$. ∎

Proposition .5. *Let $f : G \to G$ be a primitive train-track map representing an* iwip *outer automorphism Φ. The symbolic attractive lamination in G-coordinates $L^+_\Phi(G)$ is equal to the attractive lamination L_f of the double substitution of f (see Definition 2.6).*

Proof. From Definition 2.6, if f is nonorientable, L_f is defined as the symbolic set X_{σ_f} generated by σ_f. By definition, for every letter a, the two-sided sequence l_a belongs to X_{σ_f}, hence Lemma .4 implies that $L^+_\Phi(G) \subset L_f$. Conversely, if $l \in L_f$, then its language is the same as that of any periodic point of σ_f, and $l \in L^+_\Phi(G)$.

If f is orientable, the substitution (taking the square in the orientation-reversing case) splits in two disjoint primitive substitutions conjugated by Θ, and L_f is the union of the two systems associated with the two primitive substitutions. By primitivity, both subsystems are minimal and generated by any periodic point for a letter that belongs to the corresponding alphabet. Let us fix a letter a. Then l_a belongs to one of the subsystems, and $l_{\Theta(a)}$ belongs to the other one. Hence $L_f = X_{l_a} \cup X_{l_{\Theta(a)}} = X_{l_a} \cup \Theta X_{l_a} = L^+_\Phi(G)$ from Lemma .4. □

Corollary .6. *Let $f : G \to G$ be a primitive train-track map representing an* iwip *outer automorphism Φ and let S denote the shift map on the attractive lamination in G-coordinates $L^+_\Phi(G) = L_f$. Then the symbolic dynamical system (L_f, S) satisfies one of the following properties:*

- *If the double-substitution σ_f is orientable and orientation-preserving, (L_f, S) admits two subsets that are invariant by both the shift map S and the train-track map f. These subsets are minimal for the shift map. They are exchanged by the flip map Θ. This is the case of all extensions of substitutions to the free group.*
- *If the double-substitution σ_f is orientable and orientation-reversing, then (L_f, S) admits two shift-invariant subsets that are exchanged by the flip. These subsets are minimal for the shift map. The train-track f maps each invariant subset on the other.*
- *If the double substitution σ_f is nonorientable, then (L_f, S) is a minimal symbolic dynamical system.*

Appendix 2: orientable surfaces and geometric automorphisms

To conclude, we study automorphisms arising from homeomorphisms of orientable compact surfaces, and we show that they can never be *iwip* automorphisms on an odd number of letters.

The origin of the study of automorphisms of free groups can be found in the work of Nielsen, and later Thurston, on diffeomorphisms of surfaces. Indeed, such a diffeomorphism $f : M_g \to M_g$ gives rise to an automorphism of the fundamental group of the surface, which is a free group if the surface has a nonempty boundary.

To study their dynamics up to isotopy, Thurston introduced train-tracks as graphs embedded in the surface. Note that these surface train-tracks have more structure than those we have defined above: around each vertex, there are important data given by a cyclic order provided by the fact that the graph is locally planar (see [16]). For a surface train-track, one can define admissible weights, and a measurable foliation associated with admissible weights. The question of the orientability of this foliation is a particular case of the discussion above on the orientability of a train-track map.

We say that a free group automorphism is *orientably geometric* if it can be obtained by a diffeomorphism of an orientable surface M_g. Many automorphisms are not orientably geometric:

Proposition .7. *An* iwip *automorphism on a free group of odd rank is never orientably geometric.*

Proof. Suppose that f realizes an *iwip* automorphism on an orientable surface M_g. This surface must have a boundary, otherwise the fundamental group of the surface is not free. If there are $N > 1$ components of the boundary, we can suppose, taking a power if necessary, that f fixes all components of the boundary. But the conjugacy class of the subgroup generated by $N-1$ component of the boundary is a free factor, which contradicts the *iwip* hypothesis.

Hence there is exactly one component of the boundary; but in this case, it is well known that the fundamental group of the orientable surface M_g is a free group of even rank $2g$; so every orientably geometric *iwip* automorphism must be on a group of even rank. □

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